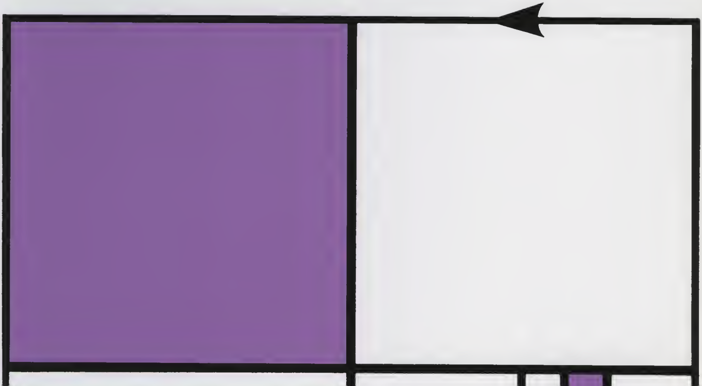
M337 Unit B4
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

UNIT B4 LAURENT SERIES



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UNIT B4 LAURENT SERIES

Prepared by the Course Team

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Before working through this text, make sure that you have read the *Course Guide* for M337 Complex Analysis.

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1.4

CONTENTS

INTRODUCTION

Cauchy's Theorem tells us that if f is a function which is analytic on a simply-connected region $\mathcal{R}$, and if Γ is a closed contour in $\mathcal{R}$, then

Unit B2, Theorem 1.2

$$\int_\Gamma f(z)\,dz = 0.$$

If, however, the function f is not analytic at some points of $\mathcal{R}$, then the conclusion of Cauchy's Theorem may fail to hold. For example, if $f(z) = z^{-1}$, and Γ is the unit circle $\{z : |z| = 1\}$, and $\mathcal{R} = \mathbb{C}$, then we have seen that

Unit B1, Example 2.2

$$\int_\Gamma z^{-1}\,dz = 2\pi i.$$

We have also seen that if f is analytic on an open disc $D = \{z : |z - \alpha| < r\}$, then f can be represented by a Taylor series

Unit B3, Theorem 3.1

$$f(z) = a_0 + a_1(z - \alpha) + \cdots + a_n(z - \alpha)^n + \cdots, \qquad \text{for } z \in D,$$

where $a_n = f^{(n)}(\alpha)/n!$. For example, if $f(z) = \sin z$ and $\alpha = 0$, then

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C}.$$

If, however, the function f is not analytic at some point α, then it is not possible to represent f as a Taylor series about α. However, as we shall see, if f is analytic on some punctured open disc $D = \{z : 0 < |z - \alpha| < r\}$ (see Figure 0.1), it is still possible to represent f by a series, although we shall need to allow negative powers of z. For example, if

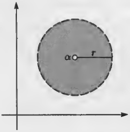


Figure 0.1 A punctured open disc

$$f(z) = \frac{\sin z}{z^3} \quad \text{and} \quad \alpha = 0,$$

then f is not analytic at 0, but it is analytic on any punctured disc with centre 0, and can be represented by the series

$$\begin{aligned}\frac{\sin z}{z^3} &= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\\ &= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}. \qquad (*)\end{aligned}$$

We say that this function f has a *singularity* at 0.

An extended power series like $(*)$, which involves negative powers of z, is called a *Laurent series*. Sometimes we need to allow infinitely many negative powers of z. For example,

$$\begin{aligned}\sin\left(\frac{1}{z}\right) &= \left(\frac{1}{z}\right) - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{5!}\left(\frac{1}{z}\right)^5 - \cdots\\ &= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}.\end{aligned}$$

In Section 1, we introduce the notion of an *isolated singularity* and show you how to classify such singularities into three types — removable singularities, poles, and essential singularities. The audio tape, which forms part of this section, contains examples of all three types of singularity.

In Section 2, we state Laurent's Theorem, which asserts that any function which is analytic on a punctured open disc (or, more generally, on an open annulus) can be represented by a Laurent series, and that this representation is unique. We then relate Laurent series to singularities.

In Section 3, we investigate the behaviour of a given function near a singularity. This behaviour turns out to be very different for the three types of singularity.

Finally, in Section 4, we focus our attention on just one of the coefficients of the Laurent series, called the *residue*, and we show how residues can be used to help us to evaluate integrals. (This is a theme which we shall return to in *Unit C1*.) We conclude with a short revision subsection, designed to test your understanding of many of the results in this block.

Study guide

Sections 1 and 2 are both lengthy, but contain material which is crucial to your understanding of the rest of the course — namely, classifying singularities (on the audio tape) and calculating Laurent series. You should try to ensure that you understand the material in these two sections before proceeding further; however, if you are short of time, you can omit Subsection 2.3 (which contains the proof of Laurent's Theorem) on a first reading.

Section 3 is largely theoretical. If you are short of time, you may prefer to read the statements of the theorems, but omit the proofs until later.

The material in Subsection 4.1 is crucial to the next block and will be reviewed at the beginning of *Unit C1*. The revision problem in Subsection 4.2 should help you to consolidate the material in this block; if you are short of time, you may prefer to work through this problem nearer to the examination.

1 SINGULARITIES

After working through this section, you should be able to:

(a) explain the term (*isolated*) *singularity*;

(b) explain what is meant by the phrase '$f(z) \to \infty$ as $z \to \alpha$';

(c) classify a given singularity as a *removable singularity*, a *pole*, or an *essential singularity*;

(d) describe the behaviour of a given function near a removable singularity or a pole.

1.1 Examples of singularities

Consider the following three functions:

$$f_1(z) = \frac{z+i}{z^2(z-2)}, \quad f_2(z) = \frac{\cos z}{\sin z}, \quad f_3(z) = z \sin\left(\frac{1}{z}\right).$$

In each case, the function is not analytic at 0, but is analytic at all points near 0, that is, on a punctured open disc with centre 0. For example,

f_1 is not analytic at 0, but is analytic on the punctured disc $\{z : 0 < |z| < 2\}$;

f_2 is not analytic at 0, but is analytic on the punctured disc $\{z : 0 < |z| < \pi\}$;

Recall that
$\sin z = 0 \iff z = 0, \pm\pi, \pm 2\pi, \ldots$.

f_3 is not analytic at 0, but is analytic on $\mathbb{C} - \{0\}$, which can be thought of as a punctured disc with infinite radius.

We say that each of these functions has an *isolated singularity* at 0.

Note that the function f_1 also has a singularity at the point 2, since f_1 is not analytic at 2, but is analytic at all points near 2; in particular, it is analytic on the punctured disc $\{z : 0 < |z-2| < 2\}$ (see Figure 1.1).

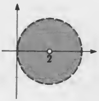


Figure 1.1

Similarly, the function f_2 has singularities at all points where $\sin z = 0$, that is, at $0, \pm\pi, \pm 2\pi, \ldots$. For example, f_2 is not analytic at 3π, but is analytic at all points near 3π; in particular, it is analytic on the punctured disc $\{z : 0 < |z - 3\pi| < \pi\}$ (see Figure 1.2).

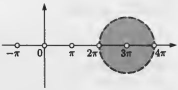


Figure 1.2

We can now give the general definition of an (isolated) singularity.

> **Definition** A function f has an **(isolated) singularity** at the point α if f is analytic on a punctured open disc $\{z : 0 < |z - \alpha| < r\}$, where $r > 0$, but not at α itself.

Just as we say 'α is a zero of f', we often say 'α is a singularity of f'.

Remarks

1 Note that the function f is usually not defined at the point α. If f is defined at α, then f is not analytic there; for example, the function

$$f(z) = \begin{cases} z, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

has a singularity at 0.

2 We usually drop the word 'isolated' and say, simply, that f has a *singularity* at α.

Example 1.1

Locate all the singularities of each of the following functions.

(a) $f(z) = \dfrac{3z}{(z-1)(z^2+4)}$ (b) $f(z) = \dfrac{e^{1/z}}{(z-2i)^2}$ (c) $f(z) = \dfrac{z+1}{\cos 2z}$

Solution

(a) The function

$$f(z) = \frac{3z}{(z-1)(z^2+4)}$$

is analytic everywhere except at the zeros of the denominator. Thus the only possible singularities are at 1, $2i$ and $-2i$. Clearly, about each of these points we can find a punctured open disc on which f is analytic — for example,

about 1, the punctured disc $\left\{z : 0 < |z - 1| < \frac{1}{2}\right\}$,
about $2i$, the punctured disc $\{z : 0 < |z - 2i| < 1\}$,
about $-2i$, the punctured disc $\{z : 0 < |z + 2i| < 1\}$,

(see Figure 1.3). Hence the singularities of f are at 1, $2i$, $-2i$.

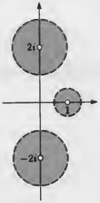


Figure 1.3

(b) The function

$$f(z) = \frac{e^{1/z}}{(z-2i)^2}$$

is analytic everywhere except at 0, where the numerator is undefined, and at $2i$, where the denominator is zero. Thus the only possible singularities are at 0 and $2i$. Clearly, about each of these points we can find a punctured open disc on which f is analytic — for example,

about 0, the punctured disc $\{z : 0 < |z| < 2\}$,
about $2i$, the punctured disc $\{z : 0 < |z - 2i| < 1\}$,

(see Figure 1.4). Hence the singularities of f are at 0, $2i$.

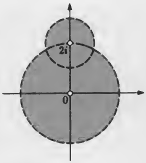


Figure 1.4

(c) The function

$$f(z) = \frac{z+1}{\cos 2z}$$

is analytic everywhere except when $\cos 2z = 0$, that is, when $2z = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$. Thus the only possible singularities are at $\pm\frac{1}{4}\pi$, $\pm\frac{3}{4}\pi$, $\pm\frac{5}{4}\pi, \ldots$. Clearly, about each of these points we can find a punctured disc (of radius $\pi/4$, say) on which f is analytic (see Figure 1.5).

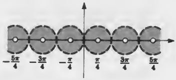


Figure 1.5

Hence the singularities of f are at $\pm\frac{1}{4}\pi, \pm\frac{3}{4}\pi, \pm\frac{5}{4}\pi, \ldots$. ■

Note that in Example 1.1 we did not always choose the largest possible punctured disc — any convenient choice will do.

Problem 1.1

Locate the singularities of each of the following functions f.

(a) $f(z) = \dfrac{z+2i}{(z-3)^2(z^2+1)}$ (b) $f(z) = \dfrac{3z-i}{z^3}\sin\left(\dfrac{1}{z+1}\right)$

(c) $f(z) = \dfrac{4e^{-z}}{z^2+2iz-1}$

Great care must be taken when identifying singularities. Consider the function

$$f(z) = \frac{1}{\sqrt{z}}.$$

At first sight, it might seem that f has a singularity at 0. However, recall that the function $z \longmapsto \sqrt{z}$ fails to be analytic at each point of the negative real axis, and so there is no punctured open disc with centre 0 on which f is analytic. Thus, f does not have a singularity at 0.

Unit A4, Example 1.4

Remember that we are using 'singularity' to mean 'isolated singularity'.

Problem 1.2

Consider the function

$$f(z) = \frac{1}{\sin(1/z)}.$$

Show that f has singularities at the points

$$\pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \ldots .$$

Explain why 0 is not a singularity of f.

1.2 Functions tending to infinity

In the next subsection, we shall describe the various types of singularity. To do this, we shall need the concept of a function tending to infinity. Consider the function

$$f(z) = 1/z^2,$$

which has a singularity at 0. If $\{z_n\}$ is any sequence of non-zero complex numbers tending to 0, then $z_n^2 \to 0$ as $n \to \infty$, and so, by the Reciprocal Rule,

$$1/z_n^2 \to \infty \text{ as } n \to \infty;$$

that is,

$$f(z_n) \to \infty \text{ as } n \to \infty.$$

This behaviour of f can be expressed as follows:

$$f(z) = 1/z^2 \to \infty \text{ as } z \to 0.$$

More generally, we give the following definition of a function *tending to infinity.*

Definition Let f be a function with domain A, and suppose that α is a limit point of A.

The function f **tends to infinity as z tends to α** if,

for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$$f(z_n) \to \infty.$$

We write $f(z) \to \infty$ as $z \to \alpha$.

We do not write

$$\lim_{z\to\infty} f(z) = \infty,$$

since this might suggest that ∞ is a complex number.

Remark The related concept

$$f(z) \to \beta \text{ as } z \to \alpha,$$

where $\alpha, \beta \in \mathbb{C}$, was introduced in *Unit A3*, Section 3. As pointed out there, this concept has an alternative ε-δ definition. In a similar way, the concept

$$f(z) \to \infty \text{ as } z \to \alpha,$$

defined above using sequences, has the following equivalent definition, which is sometimes more convenient:

for each positive number M, there is a positive number δ such that

$$|f(z)| \geq M, \qquad \text{for } z \in A \text{ and } 0 < |z - \alpha| < \delta.$$

As with sequences, there is a version of the Reciprocal Rule which relates the behaviour of functions which tend to infinity and that of functions which tend to 0.

Theorem 1.1 Reciprocal Rule

Let f be a function with domain A, and let α be a limit point of A. Then

$$f(z) \to \infty \text{ as } z \to \alpha$$

if and only if

$$\lim_{z\to\alpha} 1/f(z) = 0.$$

You are asked to prove this version of the Reciprocal Rule in Problem 1.4.

For example, taking $f(z) = 1/z^2$, we have

$$f(z) = 1/z^2 \to \infty \text{ as } z \to 0$$

because f has domain $\mathbb{C} - \{0\}$, 0 is a limit point of $\mathbb{C} - \{0\}$, and

$$\lim_{z\to 0} 1/f(z) = \lim_{z\to 0} z^2 = 0.$$

Problem 1.3

Use the Reciprocal Rule to prove that

(a) $1/(z - i) \to \infty$ as $z \to i$;

(b) $(\sin z)/z^2 \to \infty$ as $z \to 0$;

(c) $(z + i)/(z^2 + 1)^3 \to \infty$ as $z \to i$.

Problem 1.4

Prove the Reciprocal Rule. (You may assume the Reciprocal Rule for sequences.)

Unit A3, Theorem 1.5

1.3 Classifying singularities (audio-tape)

In the audio tape, you will see how to classify singularities into three types — removable singularities, poles, and essential singularities. Of these, poles will be most important for our later work, and you will see how to classify them further as poles of order 1 (simple poles), poles of order 2, and so on. We also describe the behaviour of a function near a removable singularity and near a pole. The proofs of the theorems presented here will be given in Section 3, as parts of the proofs of more general results.

NOW START THE TAPE.

1. $f(z)=(\sin z)/z, \alpha=0$

f analytic on $\mathbb{C}-\{0\}$ has a singularity at 0.

Since

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad \text{for } z \in \mathbb{C},$$

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \qquad \text{for } z \in \mathbb{C}-\{0\}.$$

Hence the function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad (z \in \mathbb{C})$$

extends f analytically to $\mathbb{C}$.

$\lim\limits_{z\to 0} f(z) = g(0) = 1$

f has a 'removable singularity' at 0.

2. $f(z)=(z+i)/(z^2+1), \alpha=-i$

f analytic on $\mathbb{C}-\{i,-i\}$ has a singularity at $-i$.

Since

$$z^2+1=(z-i)(z+i),$$

$$f(z)=\frac{1}{z-i}, \qquad \text{for } z \in \mathbb{C}-\{i,-i\}.$$

Hence the function

$$g(z)=\frac{1}{z-i} \qquad (z \in \mathbb{C}-\{i\})$$

extends f analytically to $\mathbb{C}-\{i\}$.

$\lim\limits_{z\to -i} f(z) = g(-i) = \frac{1}{2}i$

f has a 'removable singularity' at $-i$.

3. Removable singularities

f analytic on $\{z: 0<|z-\alpha|<r\}$ has a singularity at α.

The function f has a **removable singularity** at α if there is a function g, analytic at α, such that

$$f(z)=g(z), \qquad \text{for } 0<|z-\alpha|<r.$$

(g is an **analytic extension** of f to $\{z: |z-\alpha|<r\}$.)

Theorem

If f has a removable singularity at α, then

$\lim\limits_{z\to\alpha} f(z)$ exists.

$\lim\limits_{z\to\alpha} f(z) = g(\alpha)$

We may 'remove' the singularity by defining $f(\alpha)=g(\alpha)$.

4. Problem 1.5

Locate the removable singularities of each of the following functions.

(a) $f(z)=\dfrac{\sin^2 z}{z^2}$

(b) $f(z)=\dfrac{3z}{\tan z}$

(c) $f(z)=\dfrac{z^2+3iz-2}{z^2+4}$

5. $f(z) = (\sin z)/z^2$, $\alpha = 0$

f analytic on $\mathbb{C} - \{0\}$ has a singularity at 0.

Since

$$f(z) = \frac{(\sin z)/z}{z}, \qquad \text{for } z \in \mathbb{C} - \{0\},$$

we can write

$$f(z) = \frac{g(z)}{z}, \qquad \text{for } z \in \mathbb{C} - \{0\},$$

where g is the function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad (z \in \mathbb{C}).$$

Frame 1

Since $g(0) = 1 \neq 0$,

$f(z) \to \infty$ as $z \to 0$.

$f(z)$ behaves like $\frac{1}{z}$ near 0. Singularity at 0 is not removable.

f has a 'simple pole' at 0.

6. $f(z) = (z+i)/(z^2+1)$, $\alpha = i$

f analytic on $\mathbb{C} - \{i, -i\}$ has a singularity at i.

Since

$$f(z) = \frac{1}{z-i}, \qquad \text{for } z \in \mathbb{C} - \{i, -i\},$$

$z^2 + 1 = (z-i)(z+i)$

we can write

$$f(z) = \frac{g(z)}{z-i}, \qquad \text{for } z \in \mathbb{C} - \{i, -i\},$$

where g is the function

$$g(z) = 1 \qquad (z \in \mathbb{C}).$$

Hence

$f(z) \to \infty$ as $z \to i$.

Singularity at i is not removable.

f has a 'simple pole' at i.

7. Simple poles

f analytic on $\{z : 0 < |z - \alpha| < r\}$ has a singularity at α.

The function f has a **simple pole** at α if there is a function g, analytic at α with $g(\alpha) \neq 0$, such that

$$f(z) = \frac{g(z)}{z-\alpha}, \qquad \text{for } 0 < |z - \alpha| < r.$$

Theorem

If f has a simple pole at α, then

(a) $\lim_{z \to \alpha} (z-\alpha) f(z)$ exists and is non-zero;

(b) $f(z) \to \infty$ as $z \to \alpha$.

$\lim_{z\to\alpha}(z-\alpha)f(z) = g(\alpha) \neq 0$

8. Problem 1.6

Locate the simple poles of the following functions.

(a) $f(z) = \dfrac{z-2}{(z+1)(z-3i)^2}$

(b) $f(z) = \dfrac{\cos z}{z}$

(c) $f(z) = \dfrac{z}{\sin z}$

(Hint: For part (c), note that $\sin(z - k\pi) = (-1)^k \sin z$, for $k \in \mathbb{Z}$.)

9. $f(z)=(\sin z)/z^3,\ \alpha=0$

f analytic on $\mathbb{C}-\{0\}$
has a singularity at 0.

Since

$$f(z)=\frac{(\sin z)/z}{z^2}, \qquad \text{for } z\in\mathbb{C}-\{0\},$$

we can write

$$f(z)=\frac{g(z)}{z^2}, \qquad \text{for } z\in\mathbb{C}-\{0\},$$

where g is the function

$$g(z)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots \qquad (z\in\mathbb{C}).$$

Frame 1

Since $g(0)=1\neq 0$,

$f(z)\to\infty$ as $z\to 0$.

f(z) behaves like $\frac{1}{z^2}$ near 0.
Singularity at 0 is
not removable.

f has a 'pole of order 2' at 0.

10. $f(z)=(z+i)/(z^2+1)^3,\ \alpha=i$

f analytic on $\mathbb{C}-\{i,-i\}$
has a singularity at i.

Since

$$f(z)=\frac{1}{(z+i)^2}\,\frac{1}{(z-i)^3}, \qquad \text{for } z\in\mathbb{C}-\{i,-i\},$$

z^2+1
$=(z-i)(z+i)$

we can write

$$f(z)=\frac{g(z)}{(z-i)^3}, \qquad \text{for } z\in\mathbb{C}-\{i,-i\},$$

where g is the function

$$g(z)=\frac{1}{(z+i)^2} \qquad (z\in\mathbb{C}-\{-i\}).$$

Since $g(i)=-1/4\neq 0$,

$f(z)\to\infty$ as $z\to i$.

f(z) behaves like $\frac{1}{(z-i)^3}$ near i.
Singularity at i is
not removable.

f has a 'pole of order 3' at i.

11. Poles of order k, $k\geq 1$

f analytic on
$\{z: 0<|z-\alpha|<r\}$
has a singularity at α.

The function f has a **pole of order** k at α if there is a function g, analytic at α with $g(\alpha)\neq 0$, such that

$$f(z)=\frac{g(z)}{(z-\alpha)^k}, \qquad \text{for } 0<|z-\alpha|<r.$$

Theorem

If f has a pole of order k at α, then

(a) $\lim_{z\to\alpha}(z-\alpha)^k f(z)$ exists and is non-zero;

(b) $f(z)\to\infty$ as $z\to\alpha$.

$\lim_{z\to\alpha}(z-\alpha)^k f(z)$
$=g(\alpha)$
$\neq 0$

12. Problem 1.7

Locate the poles (and find their orders) of each of the following functions.

(a) $f(z)=\dfrac{z+2}{z^4(z^2-4)^3}$

(b) $f(z)=\dfrac{z}{\sin^3 z}$

13. $f(z) = \sin(1/z), \alpha = 0$

f analytic on $\mathbb{C} - \{0\}$ has a singularity at 0.

Since

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad \text{for } z \in \mathbb{C},$$

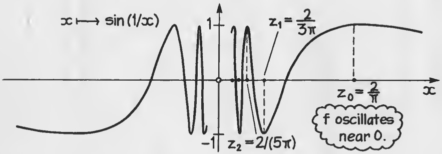
we have

$$f(z) = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}.$$

Replace z by 1/z.

The form of this series suggests that f has neither a removable singularity nor a pole at 0.

Consider the restriction of f to the real axis:



The sequence $z_n = \frac{2}{(2n+1)\pi}$, $n = 0, 1, 2, \ldots$, satisfies

$z_n \to 0$ as $n \to \infty$, but $f(z_n) = \sin((n + \frac{1}{2})\pi) = (-1)^n$.

Now

$\{f(z_n)\}$ divergent $\Longrightarrow$ singularity not removable, Frame 3

$f(z_n) \not\to \infty$ $\Longrightarrow$ singularity not a pole. Frame 11

A new type of singularity!

f has an 'essential singularity' at 0.

14. Essential singularities

f analytic on $\{z : 0 < |z - \alpha| < r\}$ has a singularity at α.

The function f has an **essential singularity** at α if the singularity at α is neither removable nor a pole.

Such a singularity cannot be removed, even if we multiply by any $(z - \alpha)^k$, where $k \geq 1$.

Hence the name 'essential'.

Theorem

If f has a singularity at α and

1. $\lim\limits_{z \to \alpha} f(z)$ does not exist,
2. $f(z)$ does not tend to infinity as $z \to \alpha$,

then α is an essential singularity of f.

15. Problem 1.8

Locate the essential singularities (if any) of each of the following functions.

(a) $f(z) = \dfrac{1}{(z-1)(z-3)^2}$ (b) $f(z) = e^{1/z}$

(c) $f(z) = \dfrac{z + i}{z(z^2 + 2iz - 1)}$

In the audio tape we introduced three types of singularity. For convenience, we collect together the definitions of these singularities here.

> **Definitions** Suppose that a function f has a singularity at α. Then
> (a) f has a **removable singularity** at α if there is a function g, analytic at α, and a positive number r, such that
> $$f(z) = g(z), \qquad \text{for } 0 < |z - \alpha| < r;$$
> (b) f has a **pole of order k** at α if there is a function g, analytic at α with $g(\alpha) \neq 0$, and a positive number r, such that
> $$f(z) = \frac{g(z)}{(z-\alpha)^k}, \qquad \text{for } 0 < |z - \alpha| < r;$$
> (c) f has an **essential singularity** at α if the singularity at α is neither removable nor a pole.

In Sections 2 and 3 we obtain alternative criteria for these three types of singularity.

2 LAURENT'S THEOREM

After working through this section, you should be able to:

(a) explain what is meant by the terms *extended power series*, *analytic part*, *singular part*, and *Laurent series*;

(b) understand the statement of Laurent's Theorem;

(c) calculate the Laurent series of a given function which is analytic on an open annulus;

(d) distinguish between the Laurent series about a removable singularity, a pole, or an essential singularity.

2.1 Statement of Laurent's Theorem

Consider the functions

$$f_1(z) = \frac{\sin z}{z^3} \quad \text{and} \quad f_2(z) = \sin\left(\frac{1}{z}\right).$$

From the audio tape we know that the functions f_1 and f_2 can be represented as series involving *negative* powers of z, as follows:

$$f_1(z) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\},$$

Frame 9 (multiply the series for g given there by $1/z^2$).

which involves only one negative power of z (namely, z^{-2}), and

$$f_2(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\},$$

Frame 13

which involves infinitely many negative powers (z^{-1}, z^{-3}, z^{-5}, ...).

Because of the presence of negative powers of z, these series are not ordinary power series. We find it useful to introduce the notion of an *extended power series*.

Definitions Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n = \cdots + \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{(z-\alpha)} + a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots, \qquad (*)$$

where $a_n \in \mathbb{C}$, for $n \in \mathbb{Z}$, is called an **extended power series about α.**

For a given z, the extended power series $(*)$ is **convergent** if the series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{-n}(z-\alpha)^{-n}$$

are both convergent; its **sum** is found by adding the sums of these two series.

Remarks

1 It is convenient to assign names to the two series in the definition. We call the first series, involving non-negative powers, the **analytic part**, as it is an ordinary power series and therefore represents an analytic function on its disc of convergence. We call the second series, involving negative powers only, the **singular part** as it is related to the singularity at α.

For example, the above series arising from the function $f_1(z) = (\sin z)/z^3$ has analytic part

$$-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$

and singular part z^{-2}; in contrast, the series arising from the function $f_2(z) = \sin(1/z)$ has analytic part 0 and singular part

$$\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots.$$

It is usual to write an extended power series with only negative powers in decreasing order of powers, as we did here.

2 Note that any ordinary power series is an extended power series with singular part 0.

3 Let $A = \{z : \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n \text{ converges}\}$. Then the function f defined by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n \qquad (z \in A)$$

is called the **sum function** of the extended power series.

Problem 2.1

Express each of the following functions as an extended power series about 0, and identify the analytic and singular parts.

(a) $f(z) = \dfrac{e^{2z}}{z^4}$ (b) $f(z) = e^z - e^{1/z}$ (c) $f(z) = \dfrac{\sin z}{z}$

Where does an extended power series converge? The clue lies in looking separately at the analytic and singular parts. The analytic part converges (and defines an analytic function) on its disc of convergence $\{z : |z-\alpha| < r\}$. The singular part can be thought of as an ordinary power series composed with $(z-\alpha)^{-1}$, and so converges when $|(z-\alpha)^{-1}| < r'$, for some number r'.

The following discussion assumes that the singular part is non-zero.

Since $|(z-\alpha)^{-1}| = |z-\alpha|^{-1}$ and

$$|z-\alpha|^{-1} < r' \iff |z-\alpha| > 1/r',$$

the singular part converges (and defines an analytic function, by the Composition Rule) on $\{z : |z-\alpha| > 1/r'\}$.

Thus both the analytic part and the singular part are convergent (and define analytic functions) on the intersection

$$A = \{z : |z-\alpha| < r\} \cap \{z : |z-\alpha| > 1/r'\}.$$

This set may take any one of the following forms:

(a) an open annulus $\quad (0 < 1/r' < r < \infty)$;

(b) a punctured open disc $\quad (1/r' = 0, r < \infty)$;

(c) a punctured plane $\quad (1/r' = 0, r = \infty)$;

(d) the 'outside' of a closed disc $\quad (0 < 1/r' < r = \infty)$;

(e) the empty set $\quad (1/r' \geq r)$.

Here we are using the statement '$r = \infty$' in the same way as in *Unit B3*, Subsection 2.1, when discussing the radius of convergence of a power series.

In each case, including case (e), the set A is an open set and is called the **annulus of convergence** of the given extended power series. Some of these cases are illustrated in Example 2.1 and Problem 2.2.

As for ordinary power series, an extended power series may converge at points of the boundary of the annulus of convergence.

Example 2.1

Find the annulus of convergence of each of the following extended power series.

(a) $\cdots + \dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{z} + 1 + \tfrac{1}{2}z + \tfrac{1}{4}z^2 + \tfrac{1}{8}z^3 + \cdots$

(b) $\cdots + \dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{z} + 1 + z + z^2 + z^3 + \cdots$

Solution

(a) The analytic part is

$$\begin{aligned} &1 + \tfrac{1}{2}z + \tfrac{1}{4}z^2 + \tfrac{1}{8}z^3 + \cdots \\ &\quad = 1 + \left(\tfrac{1}{2}z\right) + \left(\tfrac{1}{2}z\right)^2 + \left(\tfrac{1}{2}z\right)^3 + \cdots; \end{aligned}$$

this series converges when $|\tfrac{1}{2}z| < 1$, that is, for $|z| < 2$ (see Figure 2.1).

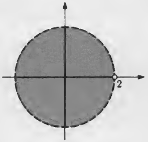


Figure 2.1

The singular part is

$$\begin{aligned} &\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \\ &\quad = \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots; \end{aligned}$$

this series converges when $0 < |1/z| < 1$, that is, for $|z| > 1$ (see Figure 2.2).

Thus the annulus of convergence of

$$\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \tfrac{1}{2}z + \tfrac{1}{4}z^2 + \tfrac{1}{8}z^3 + \cdots$$

is $\{z : 1 < |z| < 2\}$ (see Figure 2.3).

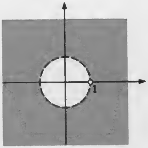


Figure 2.2

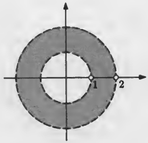


Figure 2.3

(b) The analytic part is

$$1+z+z^2+z^3+\cdots;$$

this series converges for $|z|<1$.

The singular part converges for $|z|>1$ (see part (a)).

Therefore the annulus of convergence of

$$\cdots+\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\cdots$$

is $\{z:|z|<1\}\cap\{z:|z|>1\}=\varnothing$. ■

Problem 2.2

Find the annulus of convergence of each of the following extended power series.

(a) $1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\dfrac{1}{3!z^3}+\cdots$

(b) $\dfrac{1}{z^2}+\dfrac{1}{z}+1+z+z^2+z^3+\cdots$

We remarked above that the extended power series

$$\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n$$

defines an analytic function f, say, on its annulus of convergence $A=\{z:r_1<|z-\alpha|<r_2\}$. The next theorem, which is the main result of this unit, is essentially a converse of this result. Just as an analytic function can be represented as a Taylor series on any open disc in its domain, so a function which is analytic on an annulus can be represented as an extended power series which converges at all points in the annulus.

Moreover, it can be shown that

$$f'(z)=\sum_{n=-\infty}^{\infty} na_n(z-\alpha)^{n-1}, \quad \text{for } z\in A,$$

as you would expect.

Theorem 2.1 Laurent's Theorem

If f is a function which is analytic on the open annulus

$$A=\{z:r_1<|z-\alpha|<r_2\}, \text{ where } 0\le r_1<r_2\le\infty,$$

then

$$f(z)=\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n, \qquad \text{for } z\in A, \tag{2.1}$$

where

$$a_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz, \qquad \text{for } n\in\mathbb{Z}, \tag{2.2}$$

and C is any circle with centre α lying in A.

Moreover, the representation in Equation (2.1) of f on A is unique.

We prove this theorem in Subsection 2.3.

Note the conventional use of '$r_2\le\infty$', which allows A to be, for example, a punctured plane.

The representation in Equation (2.1) of an analytic function on an open annulus is given a special name.

Definition The representation in Equation (2.1) is the **Laurent series (about α) for the function f on the annulus A.**

Pierre Alphonse Laurent (1813-1854) was a French military engineer. He discovered the theorem bearing his name in 1843, although it was known to Weierstrass in 1841.

Later in the course we often use Laurent series in the case when A is a punctured open disc. We refer to such a series as the **Laurent series about α for the function f.**

Remarks

1 Note that the form of the coefficient a_n in Equation (2.2) is exactly the same as that for the coefficient of $(z-\alpha)^n$ in a Taylor series. Here, however, it is equally valid for *negative* values of n. Recall that, for Taylor series, we have

$$a_n = f^{(n)}(\alpha)/n!, \qquad \text{for } n = 0, 1, 2, \ldots.$$

For Laurent series, however, we cannot always write a_n in this way, as f may not be differentiable at α.

2 The uniqueness statement in Laurent's Theorem is very important. It implies that *any* given representation of an analytic function f on an annulus A by an extended power series must be the Laurent series for f on A, with coefficients given by Equation (2.2). For example, if $f(z) = (\sin z)/z^3$, then we obtain the representation

$$f(z) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}, \tag{2.3}$$

by using the Taylor series for the sine function. Hence the representation in Equation (2.3) must be the Laurent series for f about 0.

3 If f is analytic on the disc $\{z : |z-\alpha| < r\}$, then the Laurent series for f on the punctured disc $\{z : 0 < |z-\alpha| < r\}$ has the same terms as the Taylor series about α for f.

4 It is important to note that a Laurent series depends on the annulus A. For example, the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

is analytic on $\mathbb{C} - \{1, 2\}$ and is thus analytic on each of the three annuli:

$$\{z : |z| < 1\}, \quad \{z : 1 < |z| < 2\}, \quad \{z : |z| > 2\},$$

illustrated in Figure 2.4. In fact, f has three different Laurent series (about 0), one on each of these annuli (see Example 2.2, below, and its preceding text, where they are calculated).

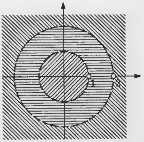


Figure 2.4

Problem 2.3

By using the Taylor series about 0 for the functions exp and sinh, write down the Laurent series about 0 for each of the following functions.

See Remark 2.

(a) $f(z) = e^{1/z}$ (b) $f(z) = \dfrac{\sinh 2z}{z^2}$

We conclude this subsection by stating a result which relates the type of singularity at α of a function f to the Laurent series about α for f. In order to introduce it, we recall three of the examples in the audio-tape section.

The function $f(z) = (\sin z)/z$ in Frame 1 has a removable singularity at 0; its Laurent series about 0 is

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}.$$

(This becomes a Taylor series if we remove the singularity by defining $f(0) = 1$, and f is then analytic on $\mathbb{C}$.)

The function $f(z) = (\sin z)/z^3$ in Frame 9 has a pole of order 2 at 0; its Laurent series about 0 is

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}.$$

The function $f(z) = \sin(1/z)$ in Frame 13 has an essential singularity at 0; its Laurent series about 0 is

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}.$$

These three functions illustrate the three parts of the following theorem.

> **Theorem 2.2** If a function f has a singularity at α and if its Laurent series about α is
>
> $$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n,$$
>
> then
>
> (a) f has a removable singularity at α if and only if $a_n = 0$ for all $n < 0$;
>
> (b) f has a pole of order k at α if and only if $a_n = 0$ for all $n < -k$, and $a_{-k} \neq 0$;
>
> (c) f has an essential singularity at α if and only if $a_n \neq 0$ for infinitely many $n < 0$.

We prove this theorem in Subsection 2.3.

Some texts use the descriptions given here as the *definitions* of these types of singularity.

To show that a function f has an essential singularity at a point by using Theorem 2.2(c) involves less work than using the theorem in Frame 14. For example, consider the function $f(z) = e^{1/z}$. The Laurent series about 0 for f is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots, \qquad \text{for } |z| > 0,$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c), f has an essential singularity at 0. This is much shorter than the solution to Problem 1.8(b), in which we used the other method.

Problem 2.4

Show that the function $f(z) = ze^{1/z}$ has an essential singularity at 0.

2.2 Calculating Laurent series

In this subsection we calculate the Laurent series for various functions which are analytic on an annulus. In the previous subsection, several Laurent series were obtained by modifying appropriate Taylor series, and this is often the easiest method. It is rarely a good idea to obtain a Laurent series by calculating the coefficients a_n using Equation (2.2); indeed, later we evaluate integrals of the type in Equation (2.2) by using the associated Laurent series.

See Problem 2.3, for example.

In Remark 4, page 18, we saw that the rational function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

is analytic on each of the three annuli

$$\{z : |z| < 1\}, \quad \{z : 1 < |z| < 2\}, \quad \{z : |z| > 2\}.$$

In order to find the three corresponding Laurent series, we start by expressing f in partial fractions:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}, \tag{2.4}$$

and then work with each term independently.

Each of these terms involves a function of the form

$$k(z) = \frac{1}{z-\beta},$$

where $\beta \neq 0$. So let us find the Laurent series about 0 for k.

The function k is analytic on $\mathbb{C} - \{\beta\}$, and so is analytic on each of the sets

$$\{z : |z| < |\beta|\} \quad \text{and} \quad \{z : |z| > |\beta|\},$$

as shown in Figure 2.5. The Laurent series about 0 for f on each of these sets is found by rearranging $1/(z-\beta)$ and using the formula for geometric series, as follows:

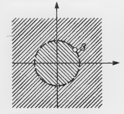


Figure 2.5

$$\begin{aligned}
k(z) &= \frac{1}{z-\beta} \\
&= -\frac{1}{\beta} \cdot \frac{1}{(1 - z/\beta)} \\
&= -\frac{1}{\beta}\left(1 + \frac{z}{\beta} + \left(\frac{z}{\beta}\right)^2 + \cdots\right), \qquad \text{for } |z/\beta| < 1, \\
&= -\frac{1}{\beta} - \frac{z}{\beta^2} - \frac{z^2}{\beta^3} - \cdots, \qquad \text{for } |z| < |\beta|; \qquad (2.5)
\end{aligned}$$

$|z/\beta| < 1 \Longleftrightarrow |z| < |\beta|$

and

$$\begin{aligned}
k(z) &= \frac{1}{z-\beta} \\
&= \frac{1}{z} \cdot \frac{1}{(1 - \beta/z)} \\
&= \frac{1}{z}\left(1 + \frac{\beta}{z} + \left(\frac{\beta}{z}\right)^2 + \cdots\right), \qquad \text{for } |\beta/z| < 1, \\
&= \frac{1}{z} + \frac{\beta}{z^2} + \frac{\beta^2}{z^3} + \cdots, \qquad \text{for } |z| > |\beta|. \qquad (2.6)
\end{aligned}$$

$|\beta/z| < 1 \Longleftrightarrow |z| > |\beta|$

(Do not attempt to remember Equations (2.5) and(2.6); rather observe their different forms.)

We now return to finding the Laurent series about 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

on $D = \{z : |z| < 1\}$.

The function $g(z) = 1/(z-2)$ is analytic on

$$G_1 = \{z : |z| < 2\} \quad \text{and} \quad G_2 = \{z : |z| > 2\};$$

the function $h(z) = 1/(z-1)$ is analytic on

$$H_1 = \{z : |z| < 1\} \quad \text{and} \quad H_2 = \{z : |z| > 1\}.$$

It is not necessary to introduce names for these sets, but this practice will help to clarify the procedure in our early examples.

Thus, since $D \subseteq G_1$ and $D \subseteq H_1$ (see Figure 2.6), we find the Laurent series for f on D, by finding the Laurent series for g on G_1 and for h on H_1, and subtracting them.

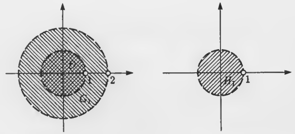


Figure 2.6 $D \subseteq G_1$, $D \subseteq H_1$

Since g is analytic on the open disc G_1 and h is analytic on the open disc H_1, their Laurent series reduce to Taylor series, as you will see.

Following the manipulations leading to Equation (2.5), we have

$$\begin{aligned}
g(z) &= \frac{1}{z-2} \\
&= -\frac{1}{2}\cdot\frac{1}{(1-z/2)} \\
&= -\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right), \qquad \text{for } |z/2|<1, \\
&= -\tfrac{1}{2}-\tfrac{1}{4}z-\tfrac{1}{8}z^2-\tfrac{1}{16}z^3-\cdots, \qquad \text{for } |z|<2;
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
h(z) &= \frac{1}{z-1} \\
&= -\frac{1}{1-z} \\
&= -\left(1+z+z^2+z^3+\cdots\right), \qquad \text{for } |z|<1, \\
&= -1-z-z^2-z^3-\cdots, \qquad \text{for } |z|<1.
\end{aligned} \tag{2.8}$$

Since Equations (2.7) and (2.8) are both valid for $|z|<1$, using Equation (2.4), we obtain

$$\begin{aligned}
f(z) &= -\tfrac{1}{2}-\tfrac{1}{4}z-\tfrac{1}{8}z^2-\tfrac{1}{16}z^3-\cdots \\
&\quad -(-1-z-z^2-z^3-\cdots) \\
&= \tfrac{1}{2}+\tfrac{3}{4}z+\tfrac{7}{8}z^2+\tfrac{15}{16}z^3+\cdots, \qquad \text{for } |z|<1.
\end{aligned}$$

We find the other two Laurent series about 0 for f in the following example.

Example 2.2

Find the Laurent series about 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

on each of the following regions.

(a) $A = \{z : 1 < |z| < 2\}$ (b) $B = \{z : |z| > 2\}$

Solution

As before, we have

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}. \tag{2.9}$$

(a) The function $g(z) = 1/(z-2)$ is analytic on

$$G_1 = \{z : |z| < 2\} \quad \text{and} \quad G_2 = \{z : |z| > 2\};$$

the function $h(z) = 1/(z-1)$ is analytic on

$$H_1 = \{z : |z| < 1\} \quad \text{and} \quad H_2 = \{z : |z| > 1\}.$$

Now $A = \{z : 1 < |z| < 2\}$ is such that

$$A \subseteq G_1 \quad \text{and} \quad A \subseteq H_2;$$

as shown in Figure 2.7.

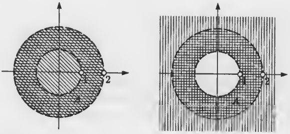


Figure 2.7 $A \subseteq G_1$, $A \subseteq H_2$

The Laurent series for g on G_1 is as in Equation (2.7):

$$g(z) = -\tfrac{1}{2} - \tfrac{1}{4}z - \tfrac{1}{8}z^2 - \tfrac{1}{16}z^3 - \cdots, \qquad \text{for } |z| < 2.$$

The Laurent series for h on H_2 is found as follows (using the manipulations which led to Equation (2.6)):

$$\begin{aligned} h(z) &= \frac{1}{z-1} \\ &= \frac{1}{z} \cdot \frac{1}{(1-1/z)} \\ &= \frac{1}{z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right), \quad \text{for } |1/z| < 1, \\ &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots, \quad \text{for } |z| > 1. \end{aligned}$$

Using Equation (2.9), we obtain

$$f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots, \quad \text{for } 1 < |z| < 2.$$

(b) Now $B = \{z : |z| > 2\}$ is such that

$$B \subseteq G_2 \quad \text{and} \quad B \subseteq H_2,$$

(see Figure 2.8).

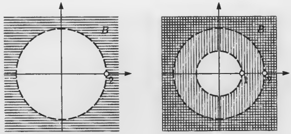


Figure 2.8 $B \subseteq G_2$, $B \subseteq H_2$

The Laurent series for g on G_2 is found as follows (using the manipulations which led to Equation (2.6)):

$$\begin{aligned} g(z) &= \frac{1}{z-2} \\ &= \frac{1}{z} \cdot \frac{1}{1-2/z} \\ &= \frac{1}{z}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right), \quad \text{for } |2/z| < 1, \\ &= \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \cdots, \quad \text{for } |z| > 2. \end{aligned}$$

The Laurent series for h on H_2 was found in part (a):

$$h(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} \cdots, \qquad \text{for } |z| > 1.$$

Using Equation (2.9), we obtain

$$f(z) = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \cdots, \quad \text{for } |z| > 2. \quad \blacksquare$$

Remarks

1 Since $f(z) = h(z)g(z)$, we could find each of the above Laurent series for f by multiplying the appropriate series for h and g. In general, this procedure is not followed since it is more tedious to multiply Laurent series than it is to add or subtract them.

2 We have combined the Laurent series for g and h term by term, just as we would do for Taylor series. Adding Laurent series in this way can be justified by adding the analytic part and singular part separately, using the Sum Rule for power series.

3 For some functions it is possible to give general terms for the analytic and singular parts of their Laurent series. So, in Example 2.2(a), we can write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(-1)z^{-n} + \sum_{n=0}^{\infty}(-1)\left(\tfrac{1}{2}\right)^{n+1} z^n \\ &= -\sum_{n=1}^{\infty} z^{-n} - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} z^n. \end{aligned}$$

We shall normally specify the general terms only if we have a use for them.

4 As you become skilled at using the partial fractions approach, you may find that you can dispense with the explicit argument relating the required annulus of convergence of f to those associated with functions appearing in the partial fractions.

Problem 2.5

(a) How many different Laurent series about 0 does the function

$$f(z) = \frac{1}{z(z-1)}$$

have?

What are their annuli of convergence?

(b) Find the Laurent series about 0 for the function f in part (a) on each of the following regions.

(i) $\{z : 0 < |z| < 1\}$ (ii) $\{z : |z| > 1\}$

(Note that there is no need to use partial fractions here.)

Problem 2.6

Find the Laurent series about 0 for the function

$$f(z) = \frac{4}{(z-1)(z+3)}$$

on each of the following regions.

(a) $\{z : |z| < 1\}$ (b) $\{z : 1 < |z| < 3\}$ (c) $\{z : |z| > 3\}$

Problem 2.7

How many different Laurent series about 0 does the function

$$f(z) = \frac{z-3i}{z(z-i)(z+3)(z-5)}$$

have?

What are their annuli of convergence?
(You are *not* required to find these Laurent series.)

Up to now, we have concentrated exclusively on Laurent series about 0. For Laurent series about other points, the simplest approach is to make a substitution which transfers the point to 0. The following example will make the method clear.

Example 2.3

Find the Laurent series about 1 for each of the following functions.

(a) $f(z) = \dfrac{e^z}{(z-1)^2}$ (b) $f(z) = \dfrac{1}{z(z-1)}$

Solution

First we note that the wording of the question means that the Laurent series we have to find converge on a punctured open disc with centre 1. The substitution $z - 1 = h$ sends each such disc to a punctured open disc with centre 0.

(a) We let $z - 1 = h$, so that $z = 1 + h$. Then, for $z \neq 1$,

$z \neq 1 \iff h \neq 0$

$$\begin{aligned} f(z) &= \frac{e^{1+h}}{h^2} \\ &= \frac{e}{h^2} \times e^h \\ &= \frac{e}{h^2}\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right), \qquad \text{for } h \in \mathbb{C} - \{0\}, \\ &= \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2} + \frac{e(z-1)}{6} + \cdots, \qquad \text{for } z \in \mathbb{C} - \{1\}. \end{aligned}$$

In this case, the series converges on the punctured plane $\mathbb{C} - \{1\}$.

(b) Again, we let $z - 1 = h$, so that $z = 1 + h$. Then, for $z \neq 0, 1$,

$$\begin{aligned} f(z) &= \frac{1}{(1+h)h} \\ &= \frac{1}{h} \cdot \frac{1}{1+h} \\ &= \frac{1}{h}(1 - h + h^2 - h^3 + \cdots), \qquad \text{for } 0 < |h| < 1, \\ &= \frac{1}{h} - 1 + h - h^2 + \cdots \\ &= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots, \qquad \text{for } 0 < |z-1| < 1. \end{aligned}$$

We choose this representation for $1/(1+h)$, because it converges on a punctured open disc with centre 0.

This series converges on the punctured disc $\{z : 0 < |z - 1| < 1\}$. ■

Problem 2.8

Find the Laurent series about 2 for each of the following functions.

(a) $f(z) = \dfrac{\cos(z-2)}{(z-2)^2}$ (b) $f(z) = z\cos\left(\dfrac{1}{z-2}\right)$ (c) $f(z) = \dfrac{1}{z^2 - 4}$

Finally, we return to the function in Example 2.2 and represent it as a Laurent series about two other points.

Example 2.4

Find the Laurent series for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

(a) about the point 1, on the punctured disc $\{z : 0 < |z - 1| < 1\}$ (see Figure 2.9);

(b) about the point $\frac{3}{2}$, on the region $\{z : |z - \frac{3}{2}| > \frac{1}{2}\}$ (see Figure 2.10).

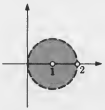


Figure 2.9

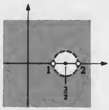


Figure 2.10

Solution

(a) Let $z-1=h$, so that $z=1+h$. Then, for $z\neq 1,2$,

The substitution $z-1=h$ sends $\{z:0<|z-1|<1\}$ to $\{h:0<|h|<1\}$.

$$\begin{aligned}
f(z) &= \frac{1}{h(h-1)}\\
&= -\frac{1}{h}\cdot\frac{1}{1-h}\\
&= -\frac{1}{h}(1+h+h^2+h^3+\cdots), \qquad \text{for } 0<|h|<1,\\
&= -\frac{1}{h}-1-h-h^2-\cdots\\
&= -\frac{1}{z-1}-1-(z-1)-(z-1)^2-\cdots, \qquad \text{for } 0<|z-1|<1.
\end{aligned}$$

This series converges on the punctured disc $\{z:0<|z-1|<1\}$.

(b) Let $z-\frac{3}{2}=h$, so that $z=\frac{3}{2}+h$. Then $z-1=h+\frac{1}{2}$ and $z-2=h-\frac{1}{2}$. Thus, for $z\neq 1,2$,

The substitution $z-\frac{3}{2}=h$ sends $\{z:|z-\frac{3}{2}|>\frac{1}{2}\}$ to $\{h:|h|>\frac{1}{2}\}$.

We *could* use partial fractions here, but it's easier not to.

$$\begin{aligned}
f(z) &= \frac{1}{(h+\frac{1}{2})(h-\frac{1}{2})}\\
&= \frac{1}{h^2-\frac{1}{4}}\\
&= \frac{1}{h^2}\cdot\frac{1}{(1-1/(4h^2))}\\
&= \frac{1}{h^2}\left(1+\left(\frac{1}{4h^2}\right)+\left(\frac{1}{4h^2}\right)^2+\left(\frac{1}{4h^2}\right)^3+\cdots\right),\\
&\qquad\qquad \text{for } |1/(4h^2)|<1,\\
&= \frac{1}{h^2}+\frac{1}{4h^4}+\frac{1}{16h^6}+\cdots, \qquad \text{for } |h|>\tfrac{1}{2},\\
&= \frac{1}{\left(z-\frac{3}{2}\right)^2}+\frac{1}{4\left(z-\frac{3}{2}\right)^4}+\frac{1}{16\left(z-\frac{3}{2}\right)^6}+\cdots, \qquad \text{for } |z-\tfrac{3}{2}|>\tfrac{1}{2}.
\end{aligned}$$

Since we are interested in the region $\{h:|h|>\frac{1}{2}\}$, we must expand in terms of $1/(4h^2)$.

This series converges on the region $\{z:|z-\frac{3}{2}|>\frac{1}{2}\}$. ■

Problem 2.9

How many different Laurent series does the function

$$f(z)=\frac{1}{z(z+3)(z+6)}$$

have

(a) about the point -3?

(b) about the point -6?

What are their annuli of convergence?

Problem 2.10

Find the Laurent series for the function

$$f(z)=\frac{4}{(z-1)(z+3)}$$

(a) about the point -3, on the punctured disc $\{z:0<|z+3|<4\}$;

(b) about the point -1, on the region $\{z:|z+1|>2\}$.

2.3 Two proofs

In this subsection we prove Theorem 2.1 (Laurent's Theorem) and Theorem 2.2.

This subsection may be omitted on a first reading.

Theorem 2.1 Laurent's Theorem

If f is a function which is analytic on the open annulus

$$A = \{z : r_1 < |z - \alpha| < r_2\}, \quad \text{where } 0 \leq r_1 \leq r_2 \leq \infty,$$

then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \qquad \text{for } z \in A, \tag{2.1}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz, \qquad \text{for } n \in \mathbb{Z}, \tag{2.2}$$

and C is any circle with centre α lying in A.

Moreover, the representation in Equation (2.1) of f on A is unique.

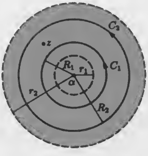


Figure 2.11

Proof The proof is in five parts.

(a) Let z be any point in A, let R_1 and R_2 be real numbers satisfying

$$r_1 < R_1 < |z - \alpha| < R_2 < r_2,$$

and let C_1 and C_2 be the circles with centre α and radii R_1 and R_2, respectively (see Figure 2.11).

We first prove that

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw. \tag{2.10}$$

To obtain this formula, we split the region between C_1 and C_2 into two parts by means of two line segments, and let Γ_1 and Γ_2 be the contours shown in Figure 2.12; the line segments are chosen so that Γ_1 encloses z. Then, by Cauchy's Integral Formula,

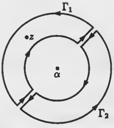


Figure 2.12

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(w)}{w-z}\,dw,$$

and, by Cauchy's Theorem,

$$0 = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(w)}{w-z}\,dw.$$

If we add these results, then the integrals along the line segments cancel, and the semicircular parts of Γ_1 and Γ_2 combine to give C_1 and C_2. This proves Equation (2.10).

(b) We now look separately at the two integrals on the right-hand side of Equation (2.10). We shall see that they give us, respectively, the analytic part and the singular part of the Laurent series (2.1).

We first note that

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} a_n(z-\alpha)^n,$$

where a_n is given by Equation (2.2).

To prove this, we simply repeat steps (b)–(d) of the proof of Taylor's Theorem in *Unit B3*; the method is identical, although we cannot deduce here that $a_n = f^{(n)}(\alpha)/n!$, as f may not be analytic at all points inside C_2. This gives the analytic part of the required series.

(c) To obtain the singular part, we consider the other integral in Equation (2.10): $-\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{f(w)}{w-z}\,dw$. First note that

$$-\frac{1}{w-z} = \frac{1}{(z-\alpha)-(w-\alpha)} = \frac{1}{z-\alpha}\left(1-\frac{w-\alpha}{z-\alpha}\right)^{-1}.$$

But, for any $\lambda \in \mathbb{C} - \{1\}$, we have

$$(1-\lambda)^{-1} = 1+\lambda+\lambda^2+\cdots+\lambda^{n-1}+\frac{\lambda^n}{1-\lambda},$$

as you can check by multiplying both sides of the equation by $1-\lambda$.

Replacing λ by $(w-\alpha)/(z-\alpha)$, and rearranging, we obtain

$$\begin{aligned}
&-\frac{1}{w-z}\\
&= \frac{1}{z-\alpha}\left(1+\frac{w-\alpha}{z-\alpha}+\cdots+\left(\frac{w-\alpha}{z-\alpha}\right)^{n-1}\right.\\
&\qquad\left.+\left(\frac{w-\alpha}{z-\alpha}\right)^{n}\cdot\frac{1}{1-(w-\alpha)/(z-\alpha)}\right)\\
&= \frac{1}{z-\alpha}+\frac{w-\alpha}{(z-\alpha)^2}+\cdots+\frac{(w-\alpha)^{n-1}}{(z-\alpha)^n}+\frac{(w-\alpha)^n}{(z-\alpha)^n}\,\frac{1}{(z-w)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw\\
&= \frac{1}{2\pi i}\int_{C_1}\left(\frac{1}{z-\alpha}+\frac{w-\alpha}{(z-\alpha)^2}+\cdots+\frac{(w-\alpha)^{n-1}}{(z-\alpha)^n}\right)f(w)\,dw\\
&\quad+\frac{1}{2\pi i}\int_{C_1}\frac{(w-\alpha)^n}{(z-\alpha)^n}\,\frac{f(w)}{z-w}\,dw.
\end{aligned}\qquad (2.11)$$

(d) Let I_n be the last integral in Equation (2.11). We now use the Estimation Theorem to show that $I_n \to 0$ as $n\to\infty$. Since f is analytic on A, f is continuous on C_1, and thus bounded on C_1; that is, there is a number K such that $|f(w)| \le K$, for $w \in C_1$. Also, by the backwards form of the Triangle Inequality,

Unit B1, Theorem 4.1

$$|z-w| \ge |z-\alpha| - |w-\alpha| = |z-\alpha| - R_1, \qquad \text{for } w \in C_1.$$

It follows from the Estimation Theorem that

$$|I_n| \le \frac{1}{2\pi}\left(\frac{R_1^n}{|z-\alpha|^n}\cdot\frac{K}{|z-\alpha|-R_1}\right)2\pi R_1 = \frac{KR_1}{|z-\alpha|-R_1}\left(\frac{R_1}{|z-\alpha|}\right)^n.$$

Since $|z-\alpha| > R_1$ (see Figure 2.11), the right-hand side tends to 0 as $n\to\infty$, so I_n tends to 0 as $n\to\infty$.

It follows from Equation (2.11) that, on letting $n\to\infty$, we have

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw\\
&= \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{z-\alpha}\,dw+\frac{1}{2\pi i}\int_{C_1}\frac{w-\alpha}{(z-\alpha)^2}f(w)\,dw\\
&\quad+\frac{1}{2\pi i}\int_{C_1}\frac{(w-\alpha)^2}{(z-\alpha)^3}f(w)\,dw+\cdots\\
&= \left(\frac{1}{2\pi i}\int_{C_1}f(w)\,dw\right)\frac{1}{z-\alpha}+\left(\frac{1}{2\pi i}\int_{C_1}(w-\alpha)f(w)\,dw\right)\frac{1}{(z-\alpha)^2}\\
&\quad+\left(\frac{1}{2\pi i}\int_{C_1}(w-\alpha)^2f(w)\,dw\right)\frac{1}{(z-\alpha)^3}+\cdots\\
&= \sum_{n=1}^{\infty} b_n(z-\alpha)^{-n}, \qquad \text{where } b_n = \frac{1}{2\pi i}\int_{C_1}(w-\alpha)^{n-1}f(w)\,dw.
\end{aligned}$$

Since $b_n = a_{-n}$ in Equation (2.2), we obtain the required singular part.

(e) Finally, we establish the uniqueness property. Suppose that

$$f(z) = \sum_{n=-\infty}^{\infty} b_n(z-\alpha)^n, \qquad \text{for } z \in A.$$

We wish to show that $a_m = b_m$ for all $m \in \mathbb{Z}$. Now, for each $m \in \mathbb{Z}$,

$$\begin{aligned}\frac{f(z)}{(z-\alpha)^{m+1}} &\\ &= \left(\sum_{n=-\infty}^{m-1} b_n(z-\alpha)^{n-m-1}\right) + \frac{b_m}{z-\alpha} + \left(\sum_{n=m+1}^{\infty} b_n(z-\alpha)^{n-m-1}\right)\\ &= f_1(z) + b_m/(z-\alpha) + f_2(z), \text{ say.} \end{aligned} \tag{2.12}$$

But the function

$$F_2(z) = \sum_{n=m+1}^{\infty} \frac{b_n}{n-m}(z-\alpha)^{n-m}$$

is a primitive of f_2 on the region $\{z : |z-\alpha| < r_2\}$, as you can check by differentiating term by term (which is permitted inside the disc of convergence). It follows from the Closed Contour Theorem that

Unit B1, Theorem 3.4

$$\int_C f_2(z)\,dz = 0,$$

where C is any circle with centre α lying in A.

Similarly, by writing $w = (z-\alpha)^{-1}$, you can check that

$$F_1(z) = \sum_{n=-\infty}^{m-1} \frac{b_n}{n-m}(z-\alpha)^{n-m}$$

Note that F_1 is obtained by composing the power series

$$\sum_{n=-\infty}^{m-1} \frac{b_n}{n-m} w^{m-n} = -\sum_{n=1-m}^{\infty} \frac{b_{-n}}{n+m} w^{m+n},$$

with $w = 1/(z-\alpha)$.

is a primitive of f_1 on the region $\{z : |z-\alpha| > r_1\}$. By the Closed Contour Theorem, therefore,

$$\int_C f_1(z)\,dz = 0.$$

Thus, by Equation (2.12),

$$a_m = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^{m+1}}\,dz = \frac{1}{2\pi i}\int_C \frac{b_m}{z-\alpha}\,dz = b_m,$$

and hence $a_m = b_m$, for all $m \in \mathbb{Z}$, as required. ∎

We now prove Theorem 2.2.

Theorem 2.2 If a function f has a singularity at α and if its Laurent series about α is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n,$$

then

(a) f has a removable singularity at α if and only if $a_n = 0$ for all $n < 0$;

(b) f has a pole of order k at α if and only if $a_n = 0$ for all $n < -k$, and $a_{-k} \neq 0$;

(c) f has an essential singularity at α if and only if $a_n \neq 0$ for infinitely many $n < 0$.

Proof

(a) If f has a removable singularity at α, then

$$f(z) = g(z), \qquad \text{for } 0 < |z - \alpha| < r,$$

where $r > 0$ and g is analytic on the disc $\{z : |z - \alpha| < r\}$. If the Taylor series about α for g is

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \qquad \text{for } |z - \alpha| < r, \tag{2.13}$$

then

$$f(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n \qquad \text{for } 0 < |z - \alpha| < r. \tag{2.14}$$

This last equation gives the Laurent series about α for f, and hence $a_n = 0$ for all $n < 0$.

Here we use the fact that the Laurent series about α for f is unique.

Conversely, if the Laurent series for f about α satisfies $a_n = 0$ for all $n < 0$, then it is of the form in Equation (2.14). Thus $f(z) = g(z)$, for $0 < |z - \alpha| < r$, where g is defined by Equation (2.13). It follows that f has a removable singularity at α.

(b) If f has a pole of order k at α, then

$$f(z) = g(z)/(z - \alpha)^k, \qquad \text{for } 0 < |z - \alpha| < r,$$

where $r > 0$ and g is analytic on the disc $\{z : |z - \alpha| < r\}$, with $g(\alpha) \neq 0$.

If the Taylor series about α for g is

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \qquad \text{for } |z - \alpha| < r, \tag{2.15}$$

then $b_0\,(= g(\alpha)) \neq 0$ and

$$f(z) = \frac{b_0}{(z - \alpha)^k} + \frac{b_1}{(z - \alpha)^{k-1}} + \cdots, \qquad \text{for } 0 < |z - \alpha| < r. \tag{2.16}$$

This last equation gives the Laurent series about α for f, and hence

$$a_n = 0 \text{ for } n < -k, \text{ and } a_{-k}\,(= b_0) \neq 0.$$

Conversely, if the Laurent series for f about α satisfies $a_n < 0$ for all $n < -k$, and $a_{-k} \neq 0$, then it is of the form in Equation (2.16). Thus

$$f(z) = g(z)/(z - \alpha)^k, \qquad \text{for } 0 < |z - \alpha| < r,$$

where g is defined by Equation (2.15). It follows that f has a pole of order k at α.

(c) If f has an essential singularity at α, then it has neither a removable singularity nor a pole at α, and thus neither of the conditions on a_n in parts (a) and (b) applies. It follows that $a_n \neq 0$ for infinitely many $n < 0$.

Conversely, if the Laurent series for f about α satisfies $a_n \neq 0$ for infinitely many $n < 0$, then f has neither a removable singularity nor a pole at α, by parts (a) and (b). It follows that f has an essential singularity at α. ■

3 BEHAVIOUR NEAR A SINGULARITY

After working through this section, you should be able to:

(a) characterize a removable singularity of a function f in terms of the behaviour of f;

(b) characterize a pole of a function f in terms of the behaviour of f;

(c) state the Casorati–Weierstrass Theorem concerning the behaviour of a function near an essential singularity.

In this section, we investigate the behaviour of an analytic function near each of the three types of singularity.

3.1 Removable singularities

Recall that a function f has a *removable singularity* at α if f is analytic on a punctured disc $\{z : 0 < |z - \alpha| < r\}$, but not at α itself, and there is a function g, analytic at α, such that

$$f(z) = g(z), \qquad \text{for } 0 < |z - \alpha| < r. \tag{3.1}$$

The following theorem gives three equivalent conditions for f to have a removable singularity at α.

Theorem 3.1 Let a function f have a singularity at α. Then the following statements are equivalent:

(A) f has a removable singularity at α;

(B) $\lim\limits_{z \to \alpha} f(z)$ exists;

(C) f is bounded on $\{z : 0 < |z - \alpha| < r\}$, for some $r > 0$;

(D) $\lim\limits_{z \to \alpha} (z - \alpha) f(z) = 0$.

The implication
$(A) \Longrightarrow (B)$
is the theorem stated in Frame 3.

Proof To prove the equivalence of the four statements, it is sufficient to show that

$$(A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (D) \Longrightarrow (A).$$

We do this in four steps.

(a) (A) $\Longrightarrow$ (B) Let g be the function, analytic at α, which satisfies Equation (3.1). Since g is analytic, and hence continuous, at α, and $f(z) = g(z)$, for $0 < |z - \alpha| < r$, we have

$$\lim_{z \to \alpha} f(z) = \lim_{z \to \alpha} g(z) = g(\alpha).$$

(b) (B) $\Longrightarrow$ (C) If we define $f(\alpha) = \lim\limits_{z \to \alpha} f(z)$, then by condition (B), f is continuous at α, and is thus bounded on the open disc $D = \{z : |z - \alpha| < r\}$, for some $r > 0$. Hence f is bounded on the punctured disc $D - \{0\}$.

(c) (C) $\Longrightarrow$ (D) Suppose that $|f(z)| \le K$, for $0 < |z - \alpha| < r$. Then

$$|(z - \alpha) f(z)| \le K|z - \alpha|, \qquad \text{for } 0 < |z - \alpha| < r,$$

and so

$$\lim_{z \to \alpha} (z - \alpha) f(z) = 0.$$

(d) (D) $\Longrightarrow$ (A) Let f be analytic on $\{z : 0 < |z - \alpha| < r\}$, say. Consider the function

$$g(z) = \begin{cases} (z-\alpha)^2 f(z), & 0 < |z - \alpha| < r, \\ 0, & z = \alpha, \end{cases}$$

which is clearly analytic on $\{z : 0 < |z - \alpha| < r\}$. We have

$$\begin{aligned} g'(\alpha) &= \lim_{z\to\alpha} \frac{g(z) - g(\alpha)}{z - \alpha} \\ &= \lim_{z\to\alpha} \frac{(z-\alpha)^2 f(z)}{z - \alpha} \\ &= \lim_{z\to\alpha} (z - \alpha) f(z) \\ &= 0, \quad \text{by (D).} \end{aligned}$$

Hence g is analytic on $\{z : |z - \alpha| < r\}$, and so can be represented by its Taylor series on this open disc. Since $g(\alpha) = g'(\alpha) = 0$, this Taylor series takes the form

$$g(z) = \sum_{m=2}^{\infty} a_m (z - \alpha)^m, \qquad \text{for } |z - \alpha| < r.$$

Hence

$$\begin{aligned} f(z) &= \sum_{m=2}^{\infty} a_m (z - \alpha)^{m-2} \\ &= \sum_{n=0}^{\infty} a_{n+2} (z - \alpha)^n, \qquad \text{for } 0 < |z - \alpha| < r. \end{aligned}$$

$m = n + 2$

Thus, by Theorem 2.2(a), f has a removable singularity at α. ■

Problem 3.1

Verify condition (D) of Theorem 3.1 for each of the following functions.

(a) $f(z) = \dfrac{\sin^2 z}{z^2}$, $\alpha = 0$ (b) $f(z) = \dfrac{3z}{\tan 3z}$, $\alpha = 0$

(c) $f(z) = \dfrac{z^2 + 3iz - 2}{z^2 + 4}$, $\alpha = -2i$

See Problem 1.5.

3.2 Poles

Recall that a function f has a *pole of order* k at α if f is analytic on a punctured disc $\{z : 0 < |z - \alpha| < r\}$, but not at α itself, and there is a function g, analytic at α with $g(\alpha) \neq 0$, such that

$$f(z) = \frac{g(z)}{(z - \alpha)^k}, \qquad \text{for } 0 < |z - \alpha| < r. \tag{3.2}$$

The following theorem gives two equivalent conditions for f to have a pole of order k at α.

Theorem 3.2 Let a function f have a singularity at α. Then the following statements are equivalent:

(A) f has a pole of order k at α;

(B) $\lim_{z\to\alpha} (z - \alpha)^k f(z)$ exists, and is non-zero;

(C) $1/f$ has a removable singularity at α which, when removed, gives rise to a zero of order k at α.

Proof To prove the equivalence of the three statements, it is sufficient to show that

$$(A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (A).$$

We do this in three steps.

(a) (A) $\Longrightarrow$ (B) Let g be the function, analytic at α with $g(\alpha) \neq 0$, which satisfies Equation (3.2). Since g is analytic, and hence continuous, at α and $f(z) = g(z)/(z-\alpha)^k$, for $0 < |z-\alpha| < r$, we have

$$\lim_{z\to\alpha} (z-\alpha)^k f(z) = \lim_{z\to\alpha} g(z) = g(\alpha),$$

which is non-zero.

(b) (B) $\Longrightarrow$ (C) Let $g(z) = (z-\alpha)^k f(z)$. Then, by Theorem 3.1, g has a removable singularity at α, which we can remove by putting $g(\alpha) = \lim_{z\to\alpha} (z-\alpha)^k f(z)$. Furthermore, since $g(\alpha) \neq 0$, there exists $r > 0$ such that $g(z) \neq 0$, for $|z-\alpha| < r$.

It follows that

$$\frac{1}{f(z)} = (z-\alpha)^k \frac{1}{g(z)}, \qquad \text{for } |z-\alpha| < r,$$

and $1/g$ is analytic at α, with $1/g(\alpha) \neq 0$.

Thus $1/f$ has a zero of order k at α.

(c) (C) $\Longrightarrow$ (A) By condition (C),

$$\frac{1}{f(z)} = (z-\alpha)^k g(z),$$

where g is analytic at α with $g(\alpha) \neq 0$. Thus there exists $r > 0$ such that $g(z) \neq 0$, for $|z-\alpha| < r$.

It follows that

$$f(z) = \frac{1/g(z)}{(z-\alpha)^k}, \qquad \text{for } 0 < |z-\alpha| < r,$$

and the function $1/g$ is analytic at α, with $1/g(\alpha) \neq 0$. Thus f has a pole of order k at α. ■

Corollary Let a function f have a singularity at α. Then f has a pole at α if and only if

$$f(z) \to \infty \text{ as } z \to \alpha.$$

Some texts take the condition '$f(z) \to \infty$ as $z \to \alpha$' as their *definition* of a pole.

Proof First assume that f has a pole at α, of order k, say. Then, by condition (B) of Theorem 3.2,

$$\lim_{z\to\alpha} (z-\alpha)^k f(z) = \lambda, \qquad \text{where } \lambda \neq 0.$$

It follows that

$$\frac{1}{f(z)} = \frac{(z-\alpha)^k}{(z-\alpha)^k f(z)} \to \frac{0}{\lambda} = 0 \text{ as } z \to \alpha.$$

Thus $f(z) \to \infty$ as $z \to \alpha$, by the Reciprocal Rule.

Conversely, suppose that $f(z) \to \infty$ as $z \to \alpha$. Then, by the Reciprocal Rule,

$$1/f(z) \to 0 \text{ as } z \to \alpha.$$

Thus the function $1/f$ has a removable singularity at α, by Theorem 3.1, which, when removed, gives rise to a zero of f. Hence f has a pole at α, by Theorem 3.2. ■

Problem 3.2

Verify condition (B) of Theorem 3.2 for each of the following functions.

(a) $f(z) = \dfrac{z+2}{z^4(z^2-4)^3}$ (b) $f(z) = \dfrac{z}{\sin^3 z}$

See Problem 1.7.

Problem 3.3

Suppose that the function f has a singularity at 0. In each of the following cases, classify this singularity.

(a) $f(z) = g(z)/z$, where g is entire and $g(0) \neq 0$.

(b) $f(z) = g(z)/z$, where g is entire and $g(0) = 0$.

(c) $f(z) = 1/g(z)$, where g is entire and has a zero of order 2 at 0.

Problem 3.4

Suppose that the function f has a pole of order m at α, and the function g has a pole of order n at α. Classify the singularities of $f + g$ and fg at α, when

(a) $m = 5, n = 3$; (b) $m = n = 4$.

3.3 Essential singularities

Recall that a singularity is an *essential singularity* if it is neither a removable singularity nor a pole. This rather negative definition might suggest that essential singularities are not very interesting. In fact, the opposite is true.

We have seen that if α is a removable singularity of f, then $f(z)$ tends to a finite limit as z tends to α. We have also seen that if α is a pole of f, then $f(z) \to \infty$ as z tends to α. What happens if α is an essential singularity?

The answer is that the function goes haywire! A celebrated result due to Picard states that, in any punctured disc with centre α, however small, f *takes all values in* $\mathbb{C}$, with at most one exception.

Emile Picard (1856–1941), a French mathematician, proved this result in 1879.

Picard's Theorem is too hard for us to prove in this course, but we can prove a related result which is itself very surprising. It is known as the Casorati–Weierstrass Theorem, and asserts that as z varies over any punctured disc with centre α, the values $f(z)$ come arbitrarily close to any given complex number w. We now state this result formally.

Theorem 3.3 Casorati–Weierstrass Theorem

Suppose that a function f has an essential singularity at α. Let w be any complex number, and let ε and δ be positive real numbers. Then there exists $z \in \mathbb{C}$ such that

$$0 < |z - \alpha| < \delta \quad \text{and} \quad |f(z) - w| < \varepsilon$$

(see Figure 3.1).

Felice Casorati (1835–1890) was professor of mathematics at Pavia. In 1864, he went to Germany for discussions with Karl Weierstrass (1815–1897), one of the founders of the theory of analytic functions. (You will meet more of Weierstrass's work in *Unit C3*.) The theorem which bears their names was published in 1869 by Casorati and in 1876 by Weierstrass.

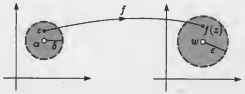


Figure 3.1

Proof Assume that the result is false. Then there exist $w \in \mathbb{C}$ and positive real numbers ε and δ such that the function f is analytic on the punctured open disc $D = \{z : 0 < |z - \alpha| < \delta\}$ and the last line of the theorem does not hold; that is,

This proof can be omitted on a first reading.

$$|f(z) - w| \geq \varepsilon, \qquad \text{for } z \in D.$$

We shall derive a contradiction from this statement.

Since $f(z) - w \neq 0$, for $z \in D$, the function

$$g(z) = 1/(f(z) - w) \qquad (z \in D)$$

is analytic. Moreover,

$$|g(z)| = 1/|f(z) - w| \leq 1/\varepsilon, \qquad \text{for } z \in D,$$

and so, by Theorem 3.1, g has a removable singularity at α. Thus, by defining $g(\alpha)$ appropriately, we can remove the singularity so that g is analytic on the disc $\{z : |z - \alpha| < \delta\}$.

Now $g(z) \neq 0$, for $z \in D$, and so

$$f(z) = w + \frac{1}{g(z)}, \qquad \text{for } z \in D.$$

If $g(\alpha) \neq 0$, then f would have a removable singularity at α, which could be removed by letting $f(\alpha) = w + (1/g(\alpha))$. If $g(\alpha) = 0$, then $g(z) = (z - \alpha)^k h(z)$, for some positive integer k, where h is analytic at α with $h(\alpha) \neq 0$. Thus

$$\begin{aligned}\lim_{z\to\alpha}(z - \alpha)^k f(z) &= \lim_{z\to\alpha}(z - \alpha)^k w + \lim_{z\to\alpha}\frac{(z - \alpha)^k}{g(z)} \\ &= 0 + \lim_{z\to\alpha} 1/h(z) \\ &= 1/h(\alpha) \neq 0,\end{aligned}$$

so that f would have a pole of order k at α, by Theorem 3.2. In neither case would f have an essential singularity at α. This contradiction proves the result. ■

Problem 3.5

Prove that there exists a complex number z such that

$$|z| < 10^{-3} \quad \text{and} \quad |e^{1/z} - 1000i| < 10^{-6}.$$

4 EVALUATING RESIDUES USING LAURENT SERIES

After working through this section, you should be able to:

(a) use Laurent's Theorem to evaluate integrals;

(b) define the *residue* of a function f with a singularity at α.

4.1 Integration and residues

Laurent's Theorem tells us that if f is a function which is analytic on the punctured disc $D = \{z : 0 < |z - \alpha| < r\}$, then the coefficient of $(z-\alpha)^n$ in the Laurent series about α for f can be expressed in the form

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^{n+1}}\,dw, \qquad \text{for } n \in \mathbb{Z},$$

where C is any circle with centre α lying in D.

As with Taylor series, the above integral expression is of little use for calculating the coefficients a_n — it is essentially of theoretical value only. In practice, we usually reverse the process, calculating the coefficients by other means and then using them to evaluate integrals by means of the formula

$$\int_C \frac{f(w)}{(w-\alpha)^{n+1}}\,dw = 2\pi i\, a_n. \tag{4.1}$$

Example 4.1

Use the Laurent series for the function $f(z) = \cos(1/z)$ to evaluate the integrals

(a) $\displaystyle\int_C w^{-5}\cos(1/w)\,dw$ (b) $\displaystyle\int_C w^5\cos(1/w)\,dw$,

where C is any circle with centre 0.

Solution

The function $f(z) = \cos(1/z)$ is analytic on $\mathbb{C} - \{0\}$, which contains any circle C with centre 0, and

$$\cos(1/z) = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}.$$

Hence

(a) $$\int_C w^{-5}\cos(1/w)\,dw = \int_C \frac{f(w)}{w^5}\,dw = 2\pi i\, a_4 = 0;$$

(b) $$\int_C w^5\cos(1/w)\,dw = \int_C \frac{f(w)}{w^{-5}}\,dw = 2\pi i\, a_{-6} = 2\pi i\left(-\frac{1}{6!}\right) = -\frac{\pi i}{360}. \quad \blacksquare$$

Problem 4.1

Evaluate the integrals

(a) $\displaystyle\int_C w^{-5}\sinh(1/w)\,dw$ (b) $\displaystyle\int_C w^4\sinh(1/w)\,dw$,

where C is any circle with centre 0.

Note that, if we take $n = -1$ in Equation (4.1), we obtain

$$\int_C f(w)\,dw = 2\pi i\, a_{-1}.$$

It follows that we can evaluate the integral on the left by calculating the appropriate coefficient a_{-1}. This turns out to be exceedingly useful in practice and, because of its importance, the coefficient a_{-1} is given a special name.

> **Definition** If a function f is analytic on a punctured disc with centre α, then a_{-1}, the coefficient of $(z-\alpha)^{-1}$ in the Laurent series about α for f, is called the **residue of f at α**, and is denoted by $\mathrm{Res}(f, \alpha)$.

Thus we have, on changing the variable of integration from w to z,

$$\int_C f(z)\,dz = 2\pi i\,\mathrm{Res}(f, \alpha), \tag{4.2}$$

where C is any circle with centre α in the punctured disc.

The following example illustrates the use of this result.

Example 4.2

Evaluate each of the following integrals.

(a) $\displaystyle\int_C \frac{1}{z-3i}\,dz$, where $C = \{z : |z-3i| = 1\}$

(b) $\displaystyle\int_C \sin(1/z)\,dz$, where $C = \{z : |z| = 5\}$

(c) $\displaystyle\int_C \frac{\sin z}{z}\,dz$, where $C = \{z : |z| = 1\}$

Solution

(a) The function $f(z) = 1/(z-3i)$ has a simple pole at $3i$, and the Laurent series about $3i$ for f is simply

$$\frac{1}{z-3i} + 0 + 0(z-3i) + 0(z-3i)^2 + \cdots, \qquad \text{for } z \in \mathbb{C} - \{3i\},$$

Once again, we are using the uniqueness property of Laurent series.

so $\mathrm{Res}(f, 3i) = 1$. Since C has centre $3i$, we obtain, by Equation (4.2),

$$\int_C \frac{1}{z-3i}\,dz = 2\pi i\,\mathrm{Res}(f, 3i) = 2\pi i.$$

(b) The function $f(z) = \sin(1/z)$ has an essential singularity at 0, and the Laurent series about 0 for f is

$$\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\},$$

so $\mathrm{Res}(f, 0) = 1$. Since C has centre 0, we obtain, by Equation (4.2),

$$\int_C \sin(1/z)\,dz = 2\pi i\,\mathrm{Res}(f, 0) = 2\pi i.$$

(c) The function $f(z) = (\sin z)/z$ has a removable singularity at 0, and the Laurent series about 0 for f is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\},$$

so $\mathrm{Res}(f, 0) = 0$. Since C has centre 0, we obtain, by Equation (4.2),

$$\int_C \frac{\sin z}{z}\,dz = 2\pi i\,\mathrm{Res}(f, 0) = 0. \quad \blacksquare$$

Remarks

1 Note that the result of part (a) could also have been obtained by using Cauchy's Integral Formula, or by parametrization, but this is not true for the result of part (b).

2 In part (c), if we define f at 0 by $f(0) = 1$, then f becomes an entire function and so, by Cauchy's Theorem,

$$\int_C f(z)\,dz = 0, \quad \text{as before.}$$

Problem 4.2

Evaluate each of the following integrals.

(a) $\displaystyle\int_C \frac{1}{(z+i)^2}\,dz$, where $C = \{z : |z+i| = 2\}$

(b) $\displaystyle\int_C \frac{\sin 2z}{z^4}\,dz$, where $C = \{z : |z| = 5\}$

In the next unit, we shall meet several methods for calculating residues, and we shall discuss other types of integral which can be evaluated by means of residues.

4.2 Revision

We conclude this block with a revision problem where you can apply the various techniques you have learned. It is a 'no-holds-barred' problem — you may use any of the methods in the block: direct calculation by parametrization; using the Fundamental Theorem of Calculus, Cauchy's Theorem or Cauchy's Formulas; Integration by Parts; or by the calculation of an appropriate residue. Often more than one method will work: the skill lies in determining an appropriate or quick method for each question.

Problem 4.3

Evaluate as many of the following integrals as you have time for, using whichever method seems most appropriate. For each integral, you should quote any results which you use, and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C e^z/z\,dz$, where $C = \{z : |z| = 3\}$

(b) $\displaystyle\int_C e^z/z\,dz$, where $C = \{z : |z-1| = \tfrac{1}{2}\}$

(c) $\displaystyle\int_C \sec^2 z\,dz$, where $C = \{z : |z - \tfrac{1}{2}\pi| = 1\}$

(d) $\displaystyle\int_C (\cosh z)/z^5\,dz$, where $C = \{z : |z| = \pi\}$

(e) $\displaystyle\int_C \frac{z + \frac{1}{2}\sin 2z}{(z-\pi/4)^2}\,dz$, where $C = \{z : |z - \tfrac{1}{4}\pi| = 1\}$

(f) $\displaystyle\int_C z\operatorname{cosec} z\,dz$, where $C = \{z : |z| = 1\}$

(g) $\displaystyle\int_C \exp(1/z^4)\,dz$, where $C = \{z : |z| = 2\}$

(h) $\displaystyle\int_C e^{1/z}\sin(1/z)\,dz$, where $C = \{z : |z| = 4\}$

(i) $\displaystyle\int_C \frac{e^z}{z^2(z-1)}\,dz$, where $C = \{z : |z| = \tfrac{1}{2}\}$

(j) $\displaystyle\int_C \frac{2\sin \pi z}{z^2-1}\,dz$, where $C = \{z : |z| = 2\}$

EXERCISES

Section 1

Exercise 1.1 Locate the singularities of each of the following functions and classify each singularity as a removable singularity, a pole (stating its order) or an essential singularity.

(a) $f(z) = \dfrac{1}{z^5(z-i)^2}$ (b) $f(z) = \dfrac{z^2+1}{z(z-i)}$ (c) $f(z) = \dfrac{\sinh z}{z^4}$

(d) $f(z) = \sinh \dfrac{1}{z}$ (e) $f(z) = \dfrac{e^z - 1}{z}$ (f) $f(z) = e^{z-1}$

(g) $f(z) = \cot z$ (h) $f(z) = \dfrac{1}{e^z - 1}$

(*Hint*: In part (d), use $i \sinh z = \sin(iz)$ and Frame 14; in part (h), use $e^z = e^{z-2k\pi i}$, for $k \in \mathbb{Z}$.)

Section 2

Exercise 2.1 Find the annulus of convergence of each of the following extended power series.

(a) $\dfrac{i}{z} + 1 - z + z^2 - z^3 + \cdots$

(b) $1 + \dfrac{1}{z} - \dfrac{1}{z^2} + \dfrac{1}{z^3} - \cdots$

Exercise 2.2 Determine the Laurent series about 0 for the function

$$f(z) = \left(\frac{1}{z} - \frac{1}{z^2}\right) \sin z.$$

State a punctured disc on which the function f is represented by this Laurent series.

Exercise 2.3 Find the Laurent series for the function

$$f(z) = \frac{1}{z(z-4)}$$

(a) about 0, on $\{z : 0 < |z| < 1\}$;
(b) about 0, on $\{z : |z| > 4\}$;
(c) about 2, on $\{z : |z-2| < 1\}$.

Exercise 2.4 Find the Laurent series for the function

$$f(z) = \frac{1}{(z-1)(z-3)}$$

(a) about 0, on $\{z : |z| < 1\}$;
(b) about 0, on $\{z : |z| > 3\}$;
(c) about 1, on $\{z : |z-1| < 2\}$.

Exercise 2.5 Show that each of the following functions has an essential singularity at 0.

(a) $f(z) = \cos \dfrac{1}{z}$ (b) $f(z) = z \sinh \dfrac{1}{z}$

Exercise 2.6 Suppose that f is an entire function and that the function g is defined by

$$g(z) = f(1/z) \qquad (z \in \mathbb{C} - \{0\}).$$

What can be deduced about the nature of f, given that g has

(a) a removable singularity at 0?

(b) a pole at 0?

(c) an essential singularity at 0?

Section 3

Exercise 3.1 Use Theorem 3.1 and the corollary to Theorem 3.2 to prove that if the function f has an essential singularity at a point α, then the function

$$f^2 = f \times f$$

has an essential singularity at α.

(You may assume that if $f^2(z) \to \infty$ as $z \to \alpha$, then $f(z) \to \infty$ as $z \to \alpha$.)

Section 4

Exercise 4.1 For each of the functions f below (all of which have been considered in earlier exercises) and the given circles C, evaluate

$$\int_C f(z)\,dz.$$

(a) $f(z) = \left(\dfrac{1}{z} - \dfrac{1}{z^2}\right)\sin z, \qquad C = \{z : |z| = 1\}$

(b) $f(z) = \dfrac{1}{z(z-4)}, \qquad C = \{z : |z| = \frac{1}{4}\}$

(c) $f(z) = \dfrac{1}{z(z-4)}, \qquad C = \{z : |z-2| = \frac{1}{2}\}$

(d) $f(z) = z \sinh \dfrac{1}{z}, \qquad C = \{z : |z| = 100\}$

Exercise 4.2 Evaluate

$$\int_C \frac{1}{z^2 - 4}\,dz, \qquad \text{where } C = \{z : |z + 2| = 3\}.$$

SOLUTIONS TO THE PROBLEMS

Section 1

1.1 (a) The function

$$f(z) = \frac{z + 2i}{(z-3)^2(z^2+1)}$$

is analytic everywhere except at the zeros of the denominator. Thus the only possible singularities are at 3, i and $-i$. About each of these points we can find a punctured disc on which f is analytic — for example,

about 3, the punctured disc $\{z : 0 < |z - 3| < 3\}$,
about i, the punctured disc $\{z : 0 < |z - i| < 2\}$,
about $-i$, the punctured disc $\{z : 0 < |z + i| < 2\}$.

Hence the singularities of f are at 3, i, $-i$.

(b) The function

$$f(z) = \frac{3z - i}{z^3} \sin\left(\frac{1}{z+1}\right)$$

is analytic everywhere except at 0 and -1. Thus the only possible singularities are at 0 and -1. About each of these points we can find a punctured disc on which the function f is analytic — for example,

about 0, the punctured disc $\{z : 0 < |z| < 1\}$,
about -1, the punctured disc $\{z : 0 < |z + 1| < 1\}$.

Hence the singularities of f are at 0, -1.

(c) The function

$$f(z) = \frac{4e^{-z}}{z^2 + 2iz - 1}$$

is analytic everywhere except at the zeros of $z^2 + 2iz - 1 = (z + i)^2$. Thus the only possible singularity is at $-i$. (Note that e^{-z} has no singularities.) About this point we can find a punctured disc on which f is analytic — for example,

$\{z : 0 < |z + i| < 100\}$.

Hence the (only) singularity of f is at $-i$.

1.2 The function

$$f(z) = 1/\sin(1/z)$$

is analytic everywhere except when $\sin(1/z) = 0$, that is, when $1/z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Thus the only possible singularities are at 0 and

$$\pm\frac{1}{\pi},\ \pm\frac{1}{2\pi},\ \pm\frac{1}{3\pi}, \ldots.$$

Although f is not defined at 0 (and thus is not analytic there), 0 is not a singularity of f because there is no punctured disc about 0 on which f is analytic. In fact, *any* punctured disc $\{z : 0 < |z| < r\}$ must contain infinitely many of the points $\pm 1/\pi, \pm 1/2\pi, \pm 1/3\pi, \ldots$.

Since f *is* analytic on each of the punctured discs

$\{z : 0 < |z - 1/k\pi| < r_k\}$,

where, for each $k \in \mathbb{Z} - \{0\}, r_k$ is the distance from $1/k\pi$ to the nearer of $1/(k+1)\pi$ and $1/(k-1)\pi$, we deduce that the singularities of f are at the points

$\pm 1/\pi,\ \pm 1/2\pi,\ \pm 1/3\pi, \ldots$.

1.3 (a) Let $A = \mathbb{C} - \{i\}$ and $\alpha = i$; then α is a limit point of A. Let $f(z) = 1/(z - i)$. Then

$$\lim_{z \to i} 1/f(z) = \lim_{z \to i}(z - i) = 0.$$

Hence, by the Reciprocal Rule,

$1/(z - i) \to \infty$ as $z \to i$.

(b) Let $A = \mathbb{C} - \{0\}$ and $\alpha = 0$; then α is a limit point of A. Let $f(z) = (\sin z)/z^2$. Then

$$\begin{aligned}\lim_{z \to 0} 1/f(z) &= \lim_{z \to 0} z^2/\sin z \\ &= \lim_{z \to 0} z\Big/\left(\frac{\sin z}{z}\right) \\ &= 0/1 = 0.\end{aligned}$$

Hence, by the Reciprocal Rule,

$(\sin z)/z^2 \to \infty$ as $z \to 0$.

(c) Let $A = \mathbb{C} - \{i\}$ and $\alpha = i$; then α is a limit point of A. Let $f(z) = (z + i)/(z^2 + 1)^3$. Then

$$\begin{aligned}\lim_{z \to i} 1/f(z) &= \lim_{z \to i}(z^2 + 1)^3/(z + i) \\ &= 0^3/2i = 0.\end{aligned}$$

Hence, by the Reciprocal Rule,

$(z + i)/(z^2 + 1)^3 \to \infty$ as $z \to i$.

1.4 Let f be a function with domain A, and let α be a limit point of A.

Assume that $f(z) \to \infty$ as $z \to \alpha$. Then, for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$f(z_n) \to \infty$ as $n \to \infty$.

By the Reciprocal Rule for sequences,

$1/f(z_n) \to 0$ as $n \to \infty$,

and thus

$$\lim_{z \to \alpha} 1/f(z) = 0.$$

Conversely, assume that $\lim_{z \to \alpha} 1/f(z) = 0$. Then, for each sequence $\{z_n\}$ in $A - \{\alpha\}$ such that $z_n \to \alpha$,

$1/f(z_n) \to 0$ as $n \to \infty$.

By the Reciprocal Rule for sequences,

$f(z_n) \to \infty$ as $n \to \infty$,

and thus

$f(z) \to \infty$ as $z \to \alpha$.

1.5 (a) The function $f(z) = (\sin^2 z)/z^2$ has a singularity at 0 and is analytic on $\mathbb{C} - \{0\}$. Also

$$\begin{aligned}f(z) &= \frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^2}{z^2} \\ &= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2, \quad \text{for } z \in \mathbb{C} - \{0\}.\end{aligned}$$

Let g be the function

$$g(z) = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2 \quad (z \in \mathbb{C}),$$

which is analytic at 0. Then

$f(z) = g(z), \quad$ for $z \in \mathbb{C} - \{0\}$,

and so f has a removable singularity at 0.

(b) The function $f(z) = 3z/(\tan z)$ has a singularity at 0 and is analytic on $D = \{z : 0 < |z| < \pi/2\}$. Also

$$f(z) = 3\cos z \Big/ \left(\frac{\sin z}{z}\right)$$
$$= \frac{3\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}, \quad \text{for } z \in D.$$

Let g be the function

$$g(z) = \frac{3\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)}{1 - \frac{z^2}{3!} + \frac{z^2}{5!} - \cdots},$$

which is analytic at 0. Then

$$f(z) = g(z), \quad \text{for } z \in D,$$

and so f has a removable singularity at 0.
(The other singularities of f at $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are not removable.)

(c) The function

$$f(z) = \frac{z^2 + 3iz - 2}{z^2 + 4} = \frac{(z + 2i)(z + i)}{(z + 2i)(z - 2i)}$$

has a singularity at $-2i$, and is analytic on $D = \{z : 0 < |z + 2i| < 4\}$. Also

$$f(z) = \frac{z + i}{z - 2i}, \quad \text{for } z \in D.$$

Let g be the function

$$g(z) = \frac{z + i}{z - 2i} \quad (z \in \mathbb{C} - \{2i\}),$$

which is analytic at $-2i$. Then

$$f(z) = g(z), \quad \text{for } z \in D,$$

and so f has a removable singularity at $-2i$.
(The other singularity of f at $2i$ is not removable.)

1.6 **(a)** The function

$$f(z) = \frac{z - 2}{(z + 1)(z - 3i)^2}$$

has singularities at -1 and $3i$, of which only -1 is a possible simple pole. Also, f is analytic on $D = \{z : 0 < |z + 1| < \sqrt{10}\}$.

Now

$$f(z) = \frac{g(z)}{z + 1}, \quad \text{for } z \in D,$$

where g is the function

$$g(z) = \frac{z - 2}{(z - 3i)^2} \quad (z \in \mathbb{C} - \{3i\}).$$

Since g is analytic at -1 and $g(-1) \neq 0$, f has a simple pole at -1.

(b) The function $f(z) = (\cos z)/z$ has a singularity at 0, and is analytic on $\mathbb{C} - \{0\}$.

Now

$$f(z) = \frac{g(z)}{z}, \quad \text{for } z \in \mathbb{C} - \{0\},$$

where g is the cosine function.

Since g is analytic at 0 and $g(0) \neq 0$, f has a simple pole at 0.

(c) The function $f(z) = z/(\sin z)$ has singularities at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$; the one at 0 is a removable singularity. Also, f is analytic on each punctured disc $D_k = \{z : 0 < |z - k\pi| < \pi\}$, where $k \in \mathbb{Z} - \{0\}$.

Let $k \in \mathbb{Z} - \{0\}$. Using the hint, we have

$$f(z) = \frac{(-1)^k z}{\sin(z - k\pi)}$$
$$= \frac{(-1)^k z}{(z - k\pi) - \frac{(z - k\pi)^3}{3!} + \frac{(z - k\pi)^5}{5!} - \cdots},$$
$$\text{for } 0 < |z - k\pi| < \pi.$$

Thus

$$f(z) = \frac{g_k(z)}{z - k\pi}, \quad \text{for } 0 < |z - k\pi| < \pi,$$

where g_k is the function

$$g_k(z) = \frac{(-1)^k z}{1 - \frac{(z - k\pi)^2}{3!} + \frac{(z - k\pi)^4}{5!} - \cdots}.$$

Since g_k is analytic at $k\pi$ and $g_k(k\pi) \neq 0$, f has a simple pole at $k\pi$, for each $k \in \mathbb{Z} - \{0\}$.

1.7 **(a)** The function

$$f(z) = \frac{z + 2}{z^4(z^2 - 4)^3} = \frac{z + 2}{z^4(z - 2)^3(z + 2)^3}$$

has singularities at $0, 2$ and -2. Also, f is analytic on the punctured discs

$$A = \{z : 0 < |z| < 2\}, \quad B = \{z : 0 < |z - 2| < 2\},$$
$$C = \{z : 0 < |z + 2| < 2\}.$$

We deal with the singularities one at a time.

First,

$$f(z) = \frac{g_A(z)}{z^4}, \quad \text{for } 0 < |z| < 2,$$

where g_A is the function

$$g_A(z) = \frac{z + 2}{(z^2 - 4)^3}.$$

Since g_A is analytic at 0 and $g(0) \neq 0$, f has a pole of order 4 at 0.

Second,

$$f(z) = \frac{1}{z^4(z - 2)^3(z + 2)^2}$$
$$= \frac{g_B(z)}{(z - 2)^3}, \quad \text{for } 0 < |z - 2| < 2,$$

where g_B is the function

$$g_B(z) = \frac{1}{z^4(z + 2)^2}.$$

Since g_B is analytic at 2 and $g_B(2) \neq 0$, f has a pole of order 3 at 2.

Third,

$$f(z) = \frac{1}{z^4(z - 2)^3(z + 2)^2}$$
$$= \frac{g_C(z)}{(z + 2)^2}, \quad \text{for } 0 < |z + 2| < 2,$$

where g_C is the function

$$g_C(z) = \frac{1}{z^4(z - 2)^3}.$$

Since g_C is analytic at -2 and $g_C(-2) \neq 0$, f has a pole of order 2 at -2.

(b) The function $f(z) = z/(\sin^3 z)$ has singularities at $0, \pm\pi, \pm 2\pi, \ldots$. Also f is analytic on each punctured disc $D_k = \{z : 0 < |z - k\pi| < \pi\}$, where $k \in \mathbb{Z}$.

First we consider the singularity at $k\pi$, where $k \in \mathbb{Z} - \{0\}$.

Since $\sin(z - k\pi) = (-1)^k \sin z$,

$$\begin{aligned} f(z) &= \frac{(-1)^{3k} z}{\sin^3(z - k\pi)} \\ &= \frac{(-1)^{3k} z}{\left((z-k\pi) - \frac{(z-k\pi)^3}{3!} + \frac{(z-k\pi)^5}{5!} - \cdots\right)^3} \\ &= \frac{g_k(z)}{(z-k\pi)^3}, \quad \text{for } 0 < |z - k\pi| < \pi, \end{aligned}$$

where g_k is the function

$$g_k(z) = \frac{(-1)^{3k} z}{\left(1 - \frac{(z-k\pi)^2}{3!} + \frac{(z-k\pi)^4}{5!} - \cdots\right)^3}.$$

Since g_k is analytic at $k\pi$ and $g_k(k\pi) \neq 0$, f has a pole of order 3 at $k\pi$, for each $k \in \mathbb{Z} - \{0\}$.

The singularity at 0 is a pole of order 2, as we now show.

We have

$$\begin{aligned} f(z) &= \frac{z}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^3} \\ &= \frac{z}{z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^3} \\ &= \frac{g_0(z)}{z^2}, \quad \text{for } 0 < |z| < \pi, \end{aligned}$$

where g_0 is the function

$$g_0(z) = \frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^3}.$$

Since g_0 is analytic at 0 and $g_0(0) \neq 0$, f has a pole of order 2 at 0.

1.8 **(a)** The function

$$f(z) = \frac{1}{(z-1)(z-3)^2}$$

has a simple pole at 1 and a pole of order 2 at 3. It has no essential singularities.

(b) The function $f(z) = e^{1/z}$ has an essential singularity at 0. We use the theorem in Frame 14 to establish this.

Consider the sequence $\{z_n\}$, where $z_n = 1/(n\pi i)$. Now $z_n \to 0$ as $n \to \infty$, and

$$e^{1/z_n} = e^{n\pi i} = (-1)^n.$$

Thus

$$\{e^{1/z_n}\} \text{ is divergent}$$

and

$$e^{1/z_n} \not\to \infty.$$

Hence $\lim\limits_{z\to 0} e^{1/z}$ does not exist and $f(z) = e^{1/z}$ does not tend to infinity as $z \to 0$. Hence, by the theorem, f has an essential singularity at 0.

(c) The function

$$f(z) = \frac{z+i}{z(z^2 + 2iz - 1)} = \frac{z+i}{z(z+i)^2}$$

has simple poles at 0 and $-i$. It has no essential singularities.

Section 2

2.1 **(a)**

$$\begin{aligned} f(z) &= \frac{1}{z^4}\left(1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \frac{(2z)^5}{5!} + \cdots\right) \\ &= z^{-4} + 2z^{-3} + \frac{4}{2!}z^{-2} + \frac{8}{3!}z^{-1} + \frac{16}{4!} + \frac{32}{5!}z + \cdots; \end{aligned}$$

analytic part: $\frac{16}{4!} + \frac{32}{5!}z + \frac{64}{6!}z^2 + \cdots$;

singular part: $z^{-4} + 2z^{-3} + \frac{4}{2!}z^{-2} + \frac{8}{3!}z^{-1}$.

(b)

$$\begin{aligned} f(z) &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) \\ &\quad - \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right) \\ &= \cdots - \frac{1}{3!z^3} - \frac{1}{2!z^2} - \frac{1}{z} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots; \end{aligned}$$

analytic part: $z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$;

singular part: $-\frac{1}{z} - \frac{1}{2!z^2} - \frac{1}{3!z^3} - \cdots$.

(c)

$$\begin{aligned} f(z) &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots; \end{aligned}$$

analytic part: $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$;

singular part: 0.

2.2 **(a)** The analytic part of this series is 1; it is defined for all $z \in \mathbb{C}$.

The singular part is

$$\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots.$$

Now

$$\begin{aligned} \left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{1}{(n+1)!z^{n+1}}\right| \Big/ \left|\frac{1}{n!z^n}\right| \\ &= \left|\frac{1}{n+1}\right|\left|\frac{1}{z}\right| \\ &\to 0 \text{ as } n \to \infty, \quad \text{for all } z \neq 0. \end{aligned}$$

Hence, by the Ratio Test, the singular part converges for all $z \neq 0$.

Thus the annulus of convergence of the given extended power series is $\mathbb{C} - \{0\}$.

(b) The singular part of this series is

$$\frac{1}{z^2} + \frac{1}{z},$$

which is defined for all $z \neq 0$.

The analytic part is

$$1 + z + z^2 + z^3 + \cdots,$$

which converges for $|z| < 1$.

Thus the annulus of convergence of the given extended power series is $\{z : 0 < |z| < 1\}$.

2.3 (a) Since $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots.$$

(b) Since $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$,

$$\frac{\sinh 2z}{z^2} = \frac{1}{z^2}\left(2z + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + \cdots\right)$$
$$= \frac{2}{z} + \frac{2^3}{3!}z + \frac{2^5}{5!}z^3 + \cdots.$$

2.4 (a) The Laurent series about 0 for the function $f(z) = ze^{1/z}$ is

$$ze^{1/z} = z\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right)$$
$$= z + 1 + \frac{1}{2!z} + \frac{1}{3!z^2} + \cdots, \quad \text{for } |z| > 0,$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c), f has an essential singularity at 0.

2.5 (a) Since the singularities of $f(z) = 1/(z(z-1))$ are at 0 and 1, f has two Laurent series about 0. The annuli of convergence are

$$\{z : 0 < |z| < 1\} \quad \text{and} \quad \{z : |z| > 1\}.$$

(b) We write

$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z}$$

and note that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots, \quad \text{for } |z| < 1,$$

and

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{(1 - 1/z)}$$
$$= -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right), \quad \text{for } |z| > 1.$$

Thus the Laurent series for f about 0 on

(i) $\{z : 0 < |z| < 1\}$ is

$$-\frac{1}{z}\left(1 + z + z^2 + z^3 + \cdots\right)$$
$$= -\frac{1}{z} - 1 - z - z^2 - \cdots;$$

(ii) $\{z : |z| > 1\}$ is

$$-\frac{1}{z}\left(-\frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^3} + \cdots)\right)$$
$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots.$$

2.6 We first express $f(z)$ in partial fractions:

$$f(z) = \frac{1}{z-1} - \frac{1}{z+3}. \qquad (1)$$

The function $g(z) = 1/(z-1)$ is analytic on

$$G_1 = \{z : |z| < 1\} \quad \text{and} \quad G_2 = \{z : |z| > 1\}.$$

The function $h(z) = 1/(z+3)$ is analytic on

$$H_1 = \{z : |z| < 3\} \quad \text{and} \quad H_2 = \{z : |z| > 3\}.$$

(a) Now $A = \{z : |z| < 1\}$ is such that

$$A \subseteq G_1 \quad \text{and} \quad A \subseteq H_1.$$

The Laurent series for g on G_1 is found as follows:

$$\frac{1}{z-1} = -\frac{1}{1-z}$$
$$= -1 - z - z^2 - z^3 - \cdots, \quad \text{for } |z| < 1.$$

The Laurent series for h on H_1 is found as follows:

$$\frac{1}{z+3} = \frac{1}{3} \cdot \frac{1}{(1 + z/3)}$$
$$= \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$
$$= \frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \cdots, \quad \text{for } |z| < 3.$$

Using Equation (1), we obtain

$$f(z) = -\frac{4}{3} - \frac{8}{9}z - \frac{28}{27}z^2 - \frac{80}{81}z^3 - \cdots, \quad \text{for } |z| < 1.$$

(b) Now $B = \{z : 1 < |z| < 3\}$ is such that

$$B \subseteq G_2 \quad \text{and} \quad B \subseteq H_1.$$

The Laurent series for g on G_2 is found as follows:

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - 1/z}$$
$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$
$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \quad \text{for } |z| > 1.$$

The Laurent series for h on H_1 is, from part (a):

$$\frac{1}{z+3} = \frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \cdots, \quad \text{for } |z| < 3.$$

Using Equation (1), we obtain

$$f(z) = \cdots + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}$$
$$- \frac{1}{3} + \frac{z}{9} - \frac{z^2}{27} + \frac{z^3}{81} - \cdots, \quad \text{for } 1 < |z| < 3.$$

(c) Now $C = \{z : |z| > 3\}$ is such that

$$C \subseteq G_2 \quad \text{and} \quad C \subseteq H_2.$$

The Laurent series for g on G_2 is, from part (b):

$$\frac{1}{z-1} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \quad \text{for } |z| > 1,$$

and the Laurent series for h on H_2 is found as follows:

$$\frac{1}{z+3} = \frac{1}{z} \cdot \frac{1}{1 + 3/z}$$
$$= \frac{1}{z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right)$$
$$= \frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \frac{27}{z^4} + \cdots, \quad \text{for } |z| > 3.$$

Using Equation (1), we obtain

$$f(z) = \frac{4}{z^2} - \frac{8}{z^3} + \frac{28}{z^4} - \cdots, \quad \text{for } |z| > 3.$$

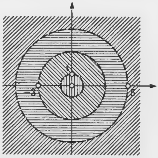
2.7 The singularities of the function

$$f(z) = \frac{z-3i}{z(z-i)(z+3)(z-5)}$$

are at 0, i, -3 and 5. Hence f has four Laurent series about 0. The annuli of convergence are

$$\{z : 0 < |z| < 1\}, \quad \{z : 1 < |z| < 3\},$$
$$\{z : 3 < |z| < 5\}, \quad \{z : |z| > 5\},$$

as shown in the figure.



2.8 (a) Let $z - 2 = h$. Then, for $z \neq 2$,

$$\begin{aligned} f(z) &= \frac{\cos h}{h^2} \\ &= \frac{1}{h^2}\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots\right), \quad \text{for } |h| > 0, \\ &= \frac{1}{h^2} - \frac{1}{2!} + \frac{h^2}{4!} - \cdots \\ &= \frac{1}{(z-2)^2} - \frac{1}{2!} + \frac{(z-2)^2}{4!} - \cdots, \quad \text{for } |z-2| > 0. \end{aligned}$$

(b) Let $z - 2 = h$, so that $z = h + 2$. Then, for $z \neq 2$,

$$\begin{aligned} f(z) &= (h+2)\cos\left(\frac{1}{h}\right) \\ &= (h+2)\left(1 - \frac{1}{2!}\left(\frac{1}{h}\right)^2 + \frac{1}{4!}\left(\frac{1}{h}\right)^4 - \cdots\right) \\ &= \left(h - \frac{1}{2!h} + \frac{1}{4!h^3} - \cdots\right) + \left(2 - \frac{2}{2!h^2} + \frac{2}{4!h^4} - \cdots\right) \\ &= h + 2 - \frac{1}{2!h} - \frac{2}{2!h^2} + \frac{1}{4!h^3} + \frac{2}{4!h^4} - \cdots, \\ &\qquad \text{for } |h| > 0, \\ &= (z-2) + 2 - \frac{1}{2(z-2)} - \frac{1}{(z-2)^2} + \frac{1}{24(z-2)^3} \\ &\quad + \frac{1}{12(z-2)^4} - \cdots, \quad \text{for } |z-2| > 0. \end{aligned}$$

(c) Let $z - 2 = h$. Then, for $z \neq 2, -2$,

$$\begin{aligned} f(z) &= \frac{1}{(z-2)(z+2)} \\ &= \frac{1}{h(h+4)} \\ &= \frac{1}{4h} \cdot \frac{1}{1 + h/4} \\ &= \frac{1}{4h}\left(1 - \left(\frac{h}{4}\right) + \left(\frac{h}{4}\right)^2 - \left(\frac{h}{4}\right)^3 + \cdots\right) \\ &= \frac{1}{4h} - \frac{1}{16} + \frac{h}{64} - \frac{h^2}{256} + \cdots, \quad \text{for } 0 < |h| < 4, \\ &= \frac{1}{4(z-2)} - \frac{1}{16} + \frac{z-2}{64} - \frac{(z-2)^2}{256} + \cdots, \\ &\qquad \text{for } 0 < |z-2| < 4. \end{aligned}$$

2.9 The function

$$f(z) = \frac{1}{z(z+3)(z+6)}$$

has singularities at 0, -3 and -6.

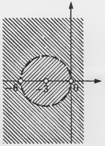
(a) f has two Laurent series about -3; the annuli of convergence are

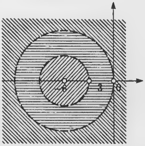
$$\{z : 0 < |z+3| < 3\} \quad \text{and} \quad \{z : |z+3| > 3\}.$$

(b) f has three Laurent series about -6; the annuli of convergence are

$$\{z : |z+6| < 3\}, \ \{z : 3 < |z+6| < 6\}, \ \{z : |z+6| > 6\}.$$

These two sets of annuli of convergence are shown in the figures.





2.10 (a) Let $z + 3 = h$, so that $z = h - 3$. Then, for $z \neq 1, -3$,

$$\begin{aligned} f(z) &= \frac{4}{(h-4)h} \\ &= -\frac{1}{h(1 - h/4)} \\ &= -\frac{1}{h} \cdot \frac{1}{1 - h/4} \\ &= -\frac{1}{h}\left(1 + \frac{h}{4} + \frac{h^2}{16} + \frac{h^3}{64} + \cdots\right), \quad \text{for } 0 < |h| < 4, \\ &= -\frac{1}{z+3} - \frac{1}{4} - \frac{z+3}{16} - \frac{(z+3)^2}{64} - \cdots, \\ &\qquad \text{for } 0 < |z+3| < 4. \end{aligned}$$

(b) Let $z + 1 = h$, so that $z = h - 1$. Then, for $z \neq 1, -3$,

$$\begin{aligned} f(z) &= \frac{4}{(h-2)(h+2)} \\ &= \frac{1}{h-2} - \frac{1}{h+2} \\ &= \frac{1}{h} \cdot \frac{1}{(1 - 2/h)} - \frac{1}{h} \cdot \frac{1}{(1 + 2/h)} \\ &= \frac{1}{h}\left(1 + \frac{2}{h} + \left(\frac{2}{h}\right)^2 + \cdots\right) \\ &\quad - \frac{1}{h}\left(1 - \frac{2}{h} + \left(\frac{2}{h}\right)^2 - \cdots\right) \\ &= \frac{4}{h^2} + \frac{16}{h^4} + \frac{64}{h^6} + \cdots, \quad \text{for } |h| > 2, \\ &= \frac{4}{(z+1)^2} + \frac{16}{(z+1)^4} + \frac{64}{(z+1)^6} + \cdots, \\ &\qquad \text{for } |z+1| > 2. \end{aligned}$$

Section 3

3.1 (a) Here $f(z) = (\sin^2 z)/z^2$, and $\alpha = 0$. We have

$$\begin{aligned}\lim_{z\to 0} z\,f(z) &= \lim_{z\to 0} z \cdot \frac{\sin^2 z}{z^2}\\ &= \lim_{z\to 0} z\left(\frac{\sin z}{z}\right)^2\\ &= 0 \times 1^2 = 0.\end{aligned}$$

Hence condition (D) is verified.

(b) Here $f(z) = 3z/(\tan 3z)$, and $\alpha = 0$. We have

$$\begin{aligned}\lim_{z\to 0} z\,f(z) &= \lim_{z\to 0} z \cdot \frac{3z}{\tan 3z}\\ &= \lim_{z\to 0} z\cos 3z\left(\frac{3z}{\sin 3z}\right)\\ &= 0 \times 1 \times 1 = 0.\end{aligned}$$

Hence condition (D) is verified.

(c) Here $f(z) = (z^2 + 3iz - 2)/(z^2 + 4)$, and $\alpha = -2i$. We have

$$\begin{aligned}\lim_{z\to -2i}(z+2i)\,f(z) &= \lim_{z\to -2i}(z+2i)\cdot\frac{z^2+3iz-2}{z^2+4}\\ &= \lim_{z\to -2i}(z+2i)\cdot\frac{(z+2i)(z+i)}{(z+2i)(z-2i)}\\ &= \lim_{z\to -2i}\frac{(z+2i)(z+i)}{z-2i}\\ &= \frac{0\times(-i)}{-4i} = 0.\end{aligned}$$

Hence condition (D) is verified.

3.2 (a) The function

$$f(z) = \frac{z+2}{z^4(z^2-4)^3}$$

has a pole of order 4 at 0, a pole of order 3 at 2 and a pole of order 2 at -2. We have

$$\begin{aligned}\lim_{z\to 0} z^4 f(z) &= \lim_{z\to 0}\frac{z+2}{(z^2-4)^3}\\ &= \frac{2}{(-4)^3} \neq 0;\end{aligned}$$

$$\begin{aligned}\lim_{z\to 2}(z-2)^3 f(z) &= \lim_{z\to 2}\frac{(z-2)^3(z+2)}{z^4(z^2-4)^3}\\ &= \lim_{z\to 2}\frac{1}{z^4(z+2)^2}\\ &= \frac{1}{2^4\cdot 4^2} \neq 0;\end{aligned}$$

$$\begin{aligned}\lim_{z\to -2}(z+2)^2 f(z) &= \lim_{z\to -2}\frac{(z+2)^3}{z^4(z^2-4)^3}\\ &= \lim_{z\to -2}\frac{1}{z^4(z-2)^3}\\ &= \frac{1}{(-2)^4\cdot(-4)^3} \neq 0.\end{aligned}$$

Hence, in each case, condition (B) is verified.

(b) The function $f(z) = z/(\sin^3 z)$ has a pole of order 2 at 0 and a pole of order 3 at $k\pi$, for each $k \in \mathbb{Z} - \{0\}$. We have

$$\begin{aligned}\lim_{z\to 0} z^2 f(z) &= \lim_{z\to 0}\frac{z^3}{\sin^3 z}\\ &= \lim_{z\to 0}\left(\frac{z}{\sin z}\right)^3\\ &= 1^3 \neq 0;\end{aligned}$$

$$\begin{aligned}\lim_{z\to k\pi}(z-k\pi)^3 f(z) &= \lim_{z\to k\pi}\frac{(z-k\pi)^3 z}{\sin^3 z}\\ &= \lim_{z\to k\pi}\frac{(-1)^{3k} z(z-k\pi)^3}{\sin^3(z-k\pi)}\\ &= \lim_{z\to k\pi}(-1)^{3k} z\left(\frac{z-k\pi}{\sin(z-k\pi)}\right)^3\\ &= (-1)^k k\pi\cdot 1^3\\ &\neq 0, \quad \text{for } k \in \mathbb{Z} - \{0\}.\end{aligned}$$

Hence, in each case, condition (B) is verified.

3.3 (a) Simple pole (by the definition in Frame 7 or Theorem 3.2, (B)).

(b) Removable singularity (by Theorem 3.1, (D)).

(c) Pole of order 2 (by Theorem 3.2, (C)).

3.4 (a) The function $f + g$ has a pole of order 5 at α, since

$$\begin{aligned}&\lim_{z\to\alpha}(z-\alpha)^5(f(z)+g(z))\\ &= \lim_{z\to\alpha}(z-\alpha)^5 f(z) + \lim_{z\to\alpha}(z-\alpha)^5 g(z)\\ &= \lim_{z\to\alpha}(z-\alpha)^5 f(z) + \lim_{z\to\alpha}(z-\alpha)^2\lim_{z\to\alpha}(z-\alpha)^3 g(z)\\ &= \lim_{z\to\alpha}(z-\alpha)^5 f(z), \quad \text{which is non-zero.}\end{aligned}$$

The result follows from condition (B) of Theorem 3.2.

The function fg has a pole of order 8 at α since

$$\begin{aligned}&\lim_{z\to\alpha}(z-\alpha)^8(fg)(z)\\ &= \left(\lim_{z\to\alpha}(z-\alpha)^5 f(z)\right)\left(\lim_{z\to\alpha}(z-\alpha)^3 g(z)\right),\end{aligned}$$

which is non-zero.

The result follows from condition (B) of Theorem 3.2.

(b) fg has a pole of order 8 at α, for reasons similar to those above. However, $f + g$ need not have a pole of order 4 at α — indeed, $f + g$ need not have a pole at all (for example, if $g = -f$); the most that can be said is that $f + g$ has a removable singularity or a pole whose order does not exceed 4.

3.5 The result follows immediately from the Casorati–Weierstrass Theorem by taking

$$f(z) = e^{1/z}, \quad \alpha = 0, \quad w = 1000i, \quad \varepsilon = 10^{-6}, \quad \delta = 10^{-3}.$$

Section 4

4.1 Let $f(z) = \sinh(1/z) = \frac{1}{z} + \frac{1}{3!z^3} + \frac{1}{5!z^5} + \cdots$. Then

(a) $\displaystyle\int_C w^{-5}\sinh(1/w)\,dw = \int_C \frac{f(w)}{w^5}\,dw$
$$= 2\pi i\, a_4 = 0;$$

(b) $\displaystyle\int_C w^4\sinh(1/w)\,dw = \int_C \frac{f(w)}{w^{-4}}\,dw$
$$= 2\pi i\, a_{-5} = 2\pi i\left(\frac{1}{5!}\right) = \frac{\pi i}{60}.$$

4.2 (a) The function $f(z) = 1/(z+i)^2$ has a pole of order 2 at the point $-i$, and the Laurent series about $-i$ for f is simply
$$\frac{1}{(z+i)^2} + \frac{0}{z+i} + 0 + \cdots, \quad \text{for } z \in \mathbb{C} - \{-i\},$$
so $\operatorname{Res}(f, -i) = 0$. Since C has centre $-i$, we obtain, by Equation (4.2),
$$\int_C \frac{1}{(z+i)^2}\,dz = 0.$$

(b) The function $f(z) = (\sin 2z)/z^4$ has a pole of order 3 at the point 0, and the Laurent series about 0 for f is
$$\frac{1}{z^4}\left((2z) - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right)$$
$$= \frac{2}{z^3} - \frac{4}{3z} + \frac{4z}{15} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$
so $\operatorname{Res}(f, 0) = -\frac{4}{3}$. Since C has centre 0, we obtain, by Equation (4.2),
$$\int_C \frac{\sin 2z}{z^4}\,dz = 2\pi i \cdot \left(-\frac{4}{3}\right) = -\frac{8}{3}\pi i.$$

4.3 In several of the following parts there are a number of ways of evaluating the integral, so do not worry if your method differs from ours.

In each part we use I to denote the relevant integral, and do not state obvious facts like '$\mathbb{C}$ is a simply-connected region' and 'the circle C is a simple-closed contour in $\mathbb{C}$', where these are relevant to the solution. Also we use the notation of named results.

(a) We use Cauchy's Integral Formula with $f(z) = e^z$, $\alpha = 0$, $\mathcal{R} = \mathbb{C}$ and $\Gamma = C$. Then f is analytic on $\mathcal{R}$, and we get
$$I = 2\pi i\, f(0) = 2\pi i e^0 = 2\pi i.$$
Alternatively, we note that the Laurent series about 0 for the function $f(z) = e^z/z$ is
$$\frac{1}{z} + 1 + \frac{z}{2!} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$
so $\operatorname{Res}(f, 0) = 1$. Since C has centre 0, we obtain, by Equation (4.2),
$$I = 2\pi i \cdot 1 = 2\pi i.$$

(b) We use Cauchy's Theorem. The function $f(z) = e^z/z$ is analytic on the simply-connected region $\mathcal{R} = \{z : \operatorname{Re} z > \frac{1}{4}\}$, which contains C. So $I = 0$.

(c) We use the Closed Contour Theorem with $f(z) = \sec^2 z$, $F(z) = \tan z$, $\mathcal{R} = \{z : 0 < |z - \frac{1}{2}\pi| < \frac{1}{2}\pi\}$ and $\Gamma = C$. Then f is continuous and has primitive F on $\mathcal{R}$, and Γ is a closed contour in $\mathcal{R}$. Hence $I = 0$.

(d) The Laurent series about 0 for the function $f(z) = (\cosh z)/z^5$ is given by
$$\frac{1}{z^5}\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right), \quad \text{for } z \in \mathbb{C} - \{0\},$$
so $\operatorname{Res}(f, 0) = 1/4!$. Since C has centre 0, we obtain, by Equation (4.2),
$$I = 2\pi i(1/4!) = \pi i/12.$$
Alternatively, we can use Cauchy's nth Derivative Formula with $n = 4$, $f(z) = \cosh z$, $\alpha = 0$, $\mathcal{R} = \mathbb{C}$ and $\Gamma = C$. Then f is analytic on $\mathcal{R}$, $f^{(4)}(0) = \cosh 0 = 1$, and we obtain
$$I = \frac{2\pi i}{4!} \cdot 1 = \pi i/12.$$

(e) We use Cauchy's First Derivative Formula with $f(z) = z + \frac{1}{2}\sin 2z$, $\alpha = \frac{1}{4}\pi$, $\mathcal{R} = \mathbb{C}$ and $\Gamma = C$. Then f is analytic on $\mathcal{R}$, $f'(\frac{1}{4}\pi) = 1 + \cos(\frac{1}{2}\pi) = 1$, and we obtain
$$I = 2\pi i \cdot 1 = 2\pi i.$$

(f) The function $f(z) = z \operatorname{cosec} z = z/(\sin z)$ has a removable singularity at 0. If we define $f(0) = \lim_{z\to 0} f(z) = 1$, then f is analytic on the simply-connected region $\mathcal{R} = \{z : |z| < \pi\}$, which contains C. Hence, by Cauchy's Theorem, $I = 0$.

(g) The Laurent series about 0 for the function $f(z) = \exp(1/z^4)$ is given by
$$1 + 1/z^4 + \frac{(1/z^4)^2}{2!} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$
so $\operatorname{Res}(f, 0) = 0$. Since C has centre 0, we obtain, by Equation (4.2),
$$I = 2\pi i \cdot 0 = 0.$$

(h) The Laurent series about 0 for the function $f(z) = e^{1/z}\sin(1/z)$ is
$$\left(1 + \frac{1}{z} + \cdots\right)\left(\frac{1}{z} - \frac{1}{3!z^3} + \cdots\right) = \frac{1}{z} + \cdots,$$
so $\operatorname{Res}(f, 0) = 1$. Since C has centre 0, we obtain, by Equation (4.2),
$$I = 2\pi i \cdot 1 = 2\pi i.$$

(i) We use Cauchy's First Derivative Formula with $f(z) = e^z/(z-1)$, $\alpha = 0$, $\mathcal{R} = \{z : \operatorname{Re} z < \frac{3}{4}\}$ and $\Gamma = C$. Then f is analytic on the simply-connected region $\mathcal{R}$, which contains C, and $f'(0) = -2$. We obtain
$$I = 2\pi i(-2) = -4\pi i.$$

(j) Since
$$\frac{1}{z^2 - 1} = \frac{1/2}{z-1} - \frac{1/2}{z+1},$$
we have
$$I = \int_C \frac{\sin \pi z}{z-1}\,dz - \int_C \frac{\sin \pi z}{z+1}\,dz.$$
We use Cauchy's Integral Formula with $f(z) = \sin \pi z$, $\alpha = 1$ and -1, $\mathcal{R} = \mathbb{C}$ and $\Gamma = C$, to evaluate the two integrals. Then f is analytic on $\mathcal{R}$, and we obtain
$$I = 2\pi i \sin(\pi) - 2\pi i \sin(-\pi) = 0.$$

SOLUTIONS TO THE EXERCISES

Section 1

1.1 **(a)** The function f has singularities at 0 and i. Also, f is analytic on $D_1 = \{z : 0 < |z| < 1\}$ and $D_2 = \{z : 0 < |z - i| < 1\}$.

Now

$$f(z) = \frac{g_1(z)}{z^5}, \quad \text{for } z \in D_1,$$

where g_1 is the function

$$g_1(z) = \frac{1}{(z-i)^2}.$$

Since g_1 is analytic at 0 and $g(0) \neq 0$, f has a pole of order 5 at 0.

Also

$$f(z) = \frac{g_2(z)}{(z-i)^2}, \quad \text{for } z \in D_2,$$

where g_2 is the function

$$g_2(z) = \frac{1}{z^5}.$$

Since g_2 is analytic at i and $g_2(i) \neq 0$, f has a pole of order 2 at i.

(b) The function f has singularities at 0 and i. Also, f is analytic on $D_1 = \{z : 0 < |z| < 1\}$ and $D_2 = \{z : 0 < |z - i| < 1\}$.

Now

$$\begin{aligned} f(z) &= \frac{(z+i)(z-i)}{z(z-i)} \\ &= \frac{z+i}{z}, \quad \text{for } z \neq 0, i, \end{aligned}$$

and so

$$f(z) = \frac{g_1(z)}{z}, \quad \text{for } z \in D_1,$$

where g_1 is the function

$$g_1(z) = z + i.$$

Since g_1 is analytic at 0 and $g_1(0) \neq 0$, f has a simple pole at 0.

Also

$$f(z) = g_2(z), \quad \text{for } z \in D_2,$$

where g_2 is the function

$$g_2(z) = \frac{z+i}{z}.$$

Since g_2 is analytic at i, f has a removable singularity at i.

(c) The function f has a singularity at 0 and is analytic on $\mathbb{C} - \{0\}$.

Now

$$f(z) = \frac{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z^4}$$

and

$$f(z) = \frac{g(z)}{z^3}, \quad \text{for } z \in \mathbb{C} - \{0\},$$

where g is the function

$$g(z) = 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots.$$

Since g is analytic at 0 and $g(0) \neq 0$, f has a pole of order 3 at 0.

(d) The function f has an essential singularity at 0. We use the theorem in Frame 14 to establish this.

Since $\sinh(1/z) = -i \sin(i/z)$, we consider the sequence $\{z_n\}$, where $z_n = 2i/((2n+1)\pi)$.

Now $z_n \to 0$ as $n \to \infty$, and

$$\begin{aligned} \sinh(1/z_n) &= -i \sin(i/z_n) \\ &= -i \sin \frac{(2n+1)\pi}{2} \\ &= -i(-1)^n. \end{aligned}$$

Thus

$$\{\sinh(1/z_n)\} \text{ is divergent}$$

and

$$\sinh(1/z_n) \not\to \infty,$$

so that $\lim_{z \to 0} \sinh(1/z)$ does not exist and $f(z) = \sinh(1/z)$ does not tend to infinity as $z \to 0$. Hence, by the theorem, f has an essential singularity at 0.

(e) The function f has a singularity at 0 and is analytic on $\mathbb{C} - \{0\}$.

Now

$$\begin{aligned} f(z) &= \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - 1}{z} \\ &= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z}, \end{aligned}$$

and so

$$f(z) = g(z), \quad \text{for } z \in \mathbb{C} - \{0\},$$

where g is the function

$$g(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots.$$

Since g is analytic at 0, f has a removable singularity at 0.

(f) The function $f(z) = e^{z-1}$ has no singularities, since it is entire.

(g) The function

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

has singularities at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Also, f is analytic on each punctured disc $D_k = \{z : 0 < |z - k\pi| < \pi\}$, where $k \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$. Since $\sin(z - k\pi) = (-1)^k \sin z$, we have

$$\begin{aligned} f(z) &= \frac{(-1)^k \cos z}{\sin(z - k\pi)} \\ &= \frac{(-1)^k \cos z}{(z - k\pi) - \frac{(z-k\pi)^3}{3!} + \frac{(z-k\pi)^5}{5!} - \cdots}, \\ &\qquad \text{for } 0 < |z - k\pi| < \pi. \end{aligned}$$

Thus

$$f(z) = \frac{g_k(z)}{z - k\pi}, \quad \text{for } 0 < |z - k\pi| < \pi,$$

where g_k is the function

$$g_k(z) = \frac{(-1)^k \cos z}{1 - \frac{(z-k\pi)^2}{3!} + \frac{(z-k\pi)^4}{5!} - \cdots}.$$

Since g_k is analytic at $k\pi$ and $g_k(k\pi) \neq 0$, f has a simple pole at $k\pi$, for each $k \in \mathbb{Z}$.

(h) The function $f(z) = 1/(e^z - 1)$ has singularities at $0, \pm 2\pi i, \pm 4\pi i, \ldots$. Also f is analytic on each punctured disc $D_k = \{z : 0 < |z - 2k\pi i| < 2\pi\}$, where $k \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$. Using the hint, we have

$$\begin{aligned} f(z) &= \frac{1}{e^{z-2k\pi i} - 1} \\ &= \frac{1}{(z - 2k\pi i) + \dfrac{(z-2k\pi i)^2}{2!} + \dfrac{(z-2k\pi i)^3}{3!} + \cdots}, \\ &\qquad\qquad \text{for } 0 < |z - 2k\pi i| < 2\pi. \end{aligned}$$

Thus

$$f(z) = \frac{g_k(z)}{z - 2k\pi i}, \quad \text{for } 0 < |z - 2k\pi i| < 2\pi,$$

where g_k is the function

$$g_k(z) = \frac{1}{1 + \dfrac{(z-2k\pi i)}{2!} + \dfrac{(z-2k\pi i)^2}{3!} + \cdots}.$$

Since g_k is analytic at $2k\pi i$ and $g_k(2k\pi i) \neq 0$, f has a simple pole at $2k\pi i$, for each $k \in \mathbb{Z}$.

Section 2

2.1 (a) The analytic part of this series is

$$1 - z + z^2 - z^3 + \cdots,$$

which converges for $|z| < 1$.

The singular part i/z is defined for all $z \neq 0$.

Thus the annulus of convergence of the given extended power series is $\{z : 0 < |z| < 1\}$.

(b) The analytic part of this series is 1, which is defined for all z.

The singular part is

$$\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots,$$

which converges for $|1/z| < 1$, that is, for $|z| > 1$.

Thus the annulus of convergence of the given extended power series is $\{z : |z| > 1\}$.

2.2 The Laurent series about 0 for the function

$$f(z) = \left(\frac{1}{z} - \frac{1}{z^2}\right)\sin z$$

is given by

$$\begin{aligned} &\left(\frac{1}{z} - \frac{1}{z^2}\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) - \left(\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots\right) \\ &= -\frac{1}{z} + 1 + \frac{z}{3!} - \frac{z^2}{3!} - \frac{z^3}{5!} + \frac{z^4}{5!} + \cdots, \quad \text{for } z \neq 0. \end{aligned}$$

The function f is represented by this Laurent series on *any* punctured disc with centre 0; for example, $\{z : 0 < |z| < 1\}$.

2.3 The function $f(z) = 1/(z(z-4))$ has two Laurent series about 0, the annuli of convergence being $A = \{z : 0 < |z| < 4\}$ and $B = \{z : |z| > 4\}$.

(a) The Laurent series about 0 for f on A converges on $D = \{z : 0 < |z| < 1\}$, since $D \subseteq A$. To determine this series, we write

$$\begin{aligned} f(z) &= \frac{1}{z(z-4)} \\ &= -\frac{1}{4z} \cdot \frac{1}{1 - z/4} \\ &= -\frac{1}{4z}\left(1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^3 + \cdots\right) \\ &= -\frac{1}{4z} - \frac{1}{16} - \frac{z}{64} - \frac{z^2}{256} - \cdots, \quad \text{for } 0 < |z| < 4. \end{aligned}$$

(b) The Laurent series about 0 for f on B is obtained as follows:

$$\begin{aligned} f(z) &= \frac{1}{z(z-4)} \\ &= \frac{1}{z^2} \cdot \frac{1}{(1 - 4/z)} \\ &= \frac{1}{z^2}\left(1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + \cdots\right) \\ &= \frac{1}{z^2} + \frac{4}{z^3} + \frac{16}{z^4} + \frac{64}{z^5} + \cdots, \quad \text{for } |z| > 4. \end{aligned}$$

(c) The function $f(z) = 1/(z(z-4))$ has two Laurent series about 2, the annuli of convergence being $A = \{z : |z-2| < 2\}$ and $B = \{z : |z-2| > 2\}$. The Laurent series about 2 for f on A converges on $D = \{z : |z-2| < 1\}$ since $D \subseteq A$. We obtain this series by using the substitution $z - 2 = h$, which sends A to $\{h : |h| < 2\}$.

Let $z - 2 = h$, so that $z = 2 + h$. Then

$$\begin{aligned} f(z) &= \frac{1}{(2+h)(h-2)} \\ &= \frac{1/4}{h-2} - \frac{1/4}{h+2} \\ &= -\frac{1}{8} \cdot \frac{1}{1 - h/2} - \frac{1}{8} \cdot \frac{1}{1 + h/2} \\ &= -\frac{1}{8}\left(1 + \frac{h}{2} + \frac{h^2}{2^2} + \frac{h^3}{2^3} + \frac{h^4}{2^4} + \cdots\right) \\ &\quad -\frac{1}{8}\left(1 - \frac{h}{2} + \frac{h^2}{2^2} - \frac{h^3}{2^3} + \frac{h^4}{2^4} - \cdots\right) \\ &= -\frac{1}{4} - \frac{h^2}{2^4} - \frac{h^4}{2^6} - \cdots, \quad \text{for } |h| < 2, \\ &= -\frac{1}{4} - \frac{(z-2)^2}{2^4} - \frac{(z-2)^4}{2^6} - \cdots, \quad \text{for } |z-2| < 2. \end{aligned}$$

2.4 We first express f in partial fractions:

$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{1/2}{z-3} \qquad (1)$$

and note that the function $g(z) = 1/(z-1)$ is analytic on

$$\{z : |z| < 1\} \quad \text{and} \quad \{z : |z| > 1\},$$

and that the function $h(z) = 1/(z-3)$ is analytic on

$$\{z : |z| < 3\} \quad \text{and} \quad \{z : |z| > 3\}.$$

(a) The Laurent series about 0 for f on $\{z : |z| < 1\}$, which is a subset of $\{z : |z| < 3\}$, is

$$\begin{aligned}f(z) &= \frac{1}{2}\cdot\frac{1}{1-z} - \frac{1}{6}\cdot\frac{1}{1-z/3}\\ &= \frac{1}{2}(1 + z + z^2 + \cdots)\\ &\quad - \frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right)\\ &= \frac{1}{3} + \frac{4}{9}z + \frac{13}{27}z^2 + \cdots, \quad \text{for } |z| < 1.\end{aligned}$$

(b) The Laurent series about 0 for f on $\{z : |z| > 3\}$, which is a subset of $\{z : |z| > 1\}$, is

$$\begin{aligned}f(z) &= -\frac{1}{2z}\cdot\frac{1}{1-1/z} + \frac{1}{2z}\cdot\frac{1}{1-3/z}\\ &= -\frac{1}{2z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)\\ &\quad + \frac{1}{2z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots\right)\\ &= \frac{1}{z^2} + \frac{4}{z^3} + \frac{13}{z^4} + \cdots, \quad \text{for } |z| > 3.\end{aligned}$$

(c) The function $f(z) = 1/((z-1)(z-3))$ has two Laurent series about 1, the annuli of convergence being $A = \{z : 0 < |z-1| < 2\}$ and $B = \{z : |z-1| > 2\}$. The Laurent series about 1 for f on A is obtained by using the substitution $z - 1 = h$, so that $z = h + 1$.

Then

$$\begin{aligned}f(z) &= \frac{1}{h(h-2)}\\ &= -\frac{1}{2h}\cdot\frac{1}{1-h/2}\\ &= -\frac{1}{2h}\left(1 + \frac{h}{2} + \frac{h^2}{2^2} + \frac{h^3}{2^3} + \cdots\right)\\ &= -\frac{1}{2h} - \frac{1}{2^2} - \frac{h}{2^3} - \frac{h^2}{2^4} - \cdots, \quad \text{for } |h| < 2,\\ &= -\frac{1}{2(z-1)} - \frac{1}{4} - \frac{(z-2)}{8} - \frac{(z-2)^2}{16} - \cdots,\\ &\qquad \text{for } |z-1| < 2.\end{aligned}$$

2.5 (a) The Laurent series about 0 for the function $f(z) = \cos(1/z)$ is

$$\cos\frac{1}{z} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \cdots, \quad \text{for } |z| > 0,$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c), f has an essential singularity at 0.

(b) The Laurent series about 0 for the function $f(z) = z\sinh(1/z)$ is

$$\begin{aligned}z\sinh\frac{1}{z} &= z\left(\frac{1}{z} + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{5!}\left(\frac{1}{z}\right)^5 + \cdots\right)\\ &= 1 + \frac{1}{3!z^2} + \frac{1}{5!z^4} + \cdots, \quad \text{for } |z| > 0,\end{aligned}$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c), f has an essential singularity at 0.

2.6 Since the function f is entire, it can be represented on $\mathbb{C}$ by its Taylor series about 0, say

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for } z \in \mathbb{C}.$$

Then

$$\begin{aligned}g(z) &= f(1/z)\\ &= \sum_{n=0}^{\infty} a_n (1/z)^n\\ &= a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.\end{aligned}$$

(a) If g has a removable singularity at 0, then, by Theorem 2.2(a),

$$0 = a_1 = a_2 = \cdots.$$

Hence $f(z) = a_0$, for $z \in \mathbb{C}$; so f is a constant function.

(b) If g has a pole of order k, say, then, by Theorem 2.2(b),

$$a_k \neq 0 \quad\text{and}\quad 0 = a_{k+1} = a_{k+2} = \cdots.$$

Hence $f(z) = a_0 + a_1 z + \cdots + a_k z^k$ is a polynomial function of degree k.

(c) If g has an essential singularity at 0, then, by Theorem 2.2(c),

$$a_k \neq 0, \quad \text{for infinitely many positive integers } k.$$

Hence $f(z) = a_0 + a_1 + a_2 z^2 + \cdots$ is an entire function, but it is not a polynomial function.

Section 3

3.1 In order to show that the function f^2 has an essential singularity at α, it is sufficient to show that f^2 has neither a removable singularity at α nor a pole there.

If f^2 has a removable singularity at α, then, by Theorem 3.1, f^2 is bounded on some punctured open disc with centre α, say

$$|f^2(z)| \leq K, \quad \text{for } 0 < |z - \alpha| < r.$$

Hence

$$|f(z)| \leq \sqrt{K}, \quad \text{for } 0 < |z - \alpha| < r,$$

so that, again by Theorem 3.1, f has a removable singularity at α, which is false.

If f^2 has a pole at α, then, by the corollary to Theorem 3.2,

$$f^2(z) \to \infty \text{ as } z \to \alpha.$$

Thus, by the given result,

$$f(z) \to \infty \text{ as } z \to \alpha,$$

and hence, again by the corollary to Theorem 3.2, f has a pole at α, which is false.

Since f^2 has neither a removable singularity nor a pole at α, it follows that f^2 has an essential singularity at α.

Section 4

4.1 (a) From Exercise 2.2, the Laurent series about 0 for the function

$$f(z) = \left(\frac{1}{z} - \frac{1}{z^2}\right)\sin z$$

is

$$-\frac{1}{z} + 1 + \frac{z}{3!} - \frac{z^2}{3!} - \cdots, \quad \text{for } z \neq 0,$$

so $\operatorname{Res}(f,0) = -1$. Since C has centre 0, we obtain, by Equation (4.2),

$$\int_C f(z)\,dz = 2\pi i \cdot (-1) = -2\pi i.$$

(b) From Exercise 2.3(a), the Laurent series about 0 on $A = \{z : 0 < |z| < 4\}$ for the function

$$f(z) = \frac{1}{z(z-4)}$$

is

$$-\frac{1}{4z} - \frac{1}{16} - \frac{z}{64} - \cdots,$$

so $\operatorname{Res}(f,0) = -\frac{1}{4}$. Since C has centre 0 and $C \subseteq A$, we obtain, by Equation (4.2),

$$\int_C f(z)\,dz = 2\pi i \cdot \left(-\tfrac{1}{4}\right) = -\tfrac{1}{2}\pi i.$$

(c) From Exercise 2.3(c), the Laurent series about 2 on $A = \{z : |z-2| < 2\}$ for the function

$$f(z) = \frac{1}{z(z-4)}$$

is

$$-\frac{1}{4} - \frac{(z-2)^2}{2^4} - \frac{(z-2)^4}{2^6} - \cdots,$$

so $\operatorname{Res}(f,2) = 0$. Since C has centre 2 and $C \subseteq A$, we obtain, by Equation (4.2),

$$\int_C f(z)\,dz = 2\pi i \cdot 0 = 0.$$

Alternatively, we simply note that the function f is analytic on A, a simply-connected region which contains the closed contour C. Hence, by Cauchy's Theorem,

$$\int_C f(z)\,dz = 0.$$

(d) From Exercise 2.5(b), the Laurent series about 0 for the function

$$f(z) = z \sinh \frac{1}{z}$$

is

$$1 + \frac{1}{3!z^2} + \frac{1}{5!z^4} + \cdots, \quad \text{for } |z| > 0,$$

so $\operatorname{Res}(f,0) = 0$. Since C has centre 0, we obtain, by Equation (4.2),

$$\int_C f(z)\,dz = 2\pi i \cdot 0 = 0.$$

4.2 The function $f(z) = 1/(z^2-4)$ is analytic on $A = \{z : 0 < |z+2| < 4\}$ and has a singularity at -2. Also C has centre -2 and $C \subseteq A$. Thus we can evaluate

$$\int_C \frac{1}{z^2-4}\,dz$$

by using Equation (4.2). To do this, we find the Laurent series about -2 for f on A.

Let $z + 2 = h$, so that $z = h - 2$. Then, for $z \neq 2, -2$,

$$\begin{aligned}
f(z) &= \frac{1}{z^2-4} \\
&= \frac{1}{(z+2)(z-2)} \\
&= \frac{1}{h(h-4)} \\
&= -\frac{1}{4h} \cdot \frac{1}{1-h/4} \\
&= -\frac{1}{4h}\left(1 + \frac{h}{4} + \frac{h^2}{16} + \cdots\right), \quad \text{for } 0 < |h| < 4, \\
&= -\frac{1}{4(z+2)} - \frac{1}{16} - \frac{(z+2)}{64} - \cdots, \\
&\qquad \text{for } 0 < |z+2| < 4.
\end{aligned}$$

So $\operatorname{Res}(f,-2) = -\frac{1}{4}$, and therefore

$$\int_C \frac{1}{z^2-4}\,dz = 2\pi i \cdot \left(-\tfrac{1}{4}\right) = -\tfrac{1}{2}\pi i.$$